

Scalar Casimir effect for D -dimensional spherically symmetric Robin boundaries

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The vacuum expectation values for the energy-momentum tensor of a massive scalar field with general curvature coupling and obeying the Robin boundary condition on spherically symmetric boundaries in D -dimensional space are investigated. The expressions are derived for the regularized vacuum energy density and radial and azimuthal stress components (i) inside and outside a single spherical surface and (ii) in the intermediate region between two concentric spheres. A regularization procedure is carried out by making use of the generalized Abel-Plana formula for the series over zeros of cylinder functions. The asymptotic behavior of the vacuum densities near the sphere and at large distances is investigated. A decomposition of the Casimir energy into volumic and surface parts is provided for both cases (i) and (ii). We show that the mode sum energy, evaluated as a sum of the zero-point energy for each normal mode of frequency, and the volume integral of the energy density in general are different, and argue that this difference is due to the existence of an additional surface energy contribution.

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1 Introduction

Historically, the investigation of the Casimir effect (for a general introduction, see [1, 2, 3]) for a perfectly conducting spherical shell was motivated by the Casimir semiclassical model of an electron. In this model Casimir suggested that Poincare stress, to stabilize the charged particle, could arise from vacuum quantum fluctuations and the fine structure constant can be determined by a balance between the Casimir force (assumed attractive) and the Coulomb repulsion. However, as has been shown by Boyer [4], the Casimir energy for the sphere is positive, implying a repulsive force. This result was later reconsidered by a number of authors [5, 6, 7]. More recently new methods have been developed for this problem including direct mode summation techniques based on the zeta function regularization scheme [8]-[20] (for similar considerations in the case of a dielectric ball see, for instance, the references given in [21]).

Investigation of the dimensional dependence of physical quantities in the Casimir effect is of considerable interest. In particular, for periodic boundary conditions this is motivated by the idea of using the Casimir effect as a source for dynamical compactification of the extra dimensions in Kaluza-Klein models. In [22] the Casimir energy is derived in a general hypercuboidal region and for various types of boundary condition and field. For a spherical shell the

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Casimir effect in an arbitrary number of dimensions is analyzed in [9, 10] for a massless scalar field satisfying Dirichlet and a special type of Robin (corresponding to the electromagnetic TM modes) boundary conditions using the Green's function method (see also [23]) and in [19] for the electromagnetic field and massless scalar and spinor fields (for the 4D fermionic Casimir effect see [24, 25, 26]) with various boundary conditions on the basis of the zeta regularization technique.

However, the most of the previous studies on spherical geometry were focused on global quantities such as the total energy and the force acting on a shell. Investigation of the energy distribution inside a perfectly reflecting spherical shell was made in [27] in the case of QED and in [28] for QCD. The distribution of the other components for the energy-momentum tensor of the electromagnetic field inside as well as outside the shell can be obtained from the results of [29, 30]. In these papers the consideration was carried out in terms of the Green's function (for this formalism see [7]). Another efficient method for extracting finite parts from the vacuum expectation values of the local physical observables in the case of plane boundaries is based on the Abel-Plana summation formula (see, e.g., [1]). We have generalized this formula to include curved boundaries [31, 32]. In [33, 34] (see also [32]) the calculations of the regularized vacuum expectation values for the electromagnetic energy-momentum tensor inside and outside a perfectly conducting spherical shell and in the region between two concentric spheres are based on the generalized Abel-Plana formula.

In this paper the vacuum expectation values of the energy-momentum tensor are investigated for a massive scalar field with general curvature coupling parameter ξ , satisfying the Robin boundary condition on spherically symmetric boundaries in D -dimensional space. As special cases they include the results for the Dirichlet, Neumann, TM, and conformally invariant Hawking boundary conditions. Robin type conditions also appear in considerations of the vacuum effects for a confined charged scalar field in external fields (see, for instance, [35]) and in quantum gravity [36]. In addition to describing the physical structure of the quantum field at a given point, the energy-momentum tensor acts as the source of gravity in the Einstein equations. It therefore plays an important role in modeling a self-consistent dynamics involving the gravitational field [37]. To the author's knowledge all previous investigations on the spherical Casimir effect for scalar fields were concerned mainly with global properties (interior and exterior energies, vacuum stress on a sphere). The massless scalar field with Dirichlet boundary condition is considered in [38, 11, 12, 9, 16, 19] and for Neumann and Robin boundary conditions in [12, 10, 16, 19]. The case of the massive scalar field is investigated in [15] (for the heat-kernel coefficients and determinants, see [13, 14] and references therein). In [39, 40] asymptotic expansions for the renormalized energy-momentum tensor are developed near an arbitrary smooth boundary in the case of conformally and minimally coupled 4D massless scalars with Dirichlet, Neumann, and Robin boundary conditions. Here we consider the scalar vacuum inside and outside a single spherical shell, and in the intermediate region between two concentric spherical shells. For the latter geometry the general case is investigated when the constants in the Robin boundary condition are different for inner and outer spheres. To evaluate the corresponding field products we use the mode sum method in combination with the summation formulas from Ref. [31] (see also [32]). For scalars with general curvature coupling the essential point is the relation between the mode sum energy, evaluated as a renormalized sum of the zero-point energies for each normal mode of frequency, and the volume integral of the renormalised energy density. For flat spacetime backgrounds the first quantity does not depend on ξ , since the normal modes are the same for fields with different values of this parameter (for instance, for minimal and conformal couplings). Nevertheless, the corresponding energy-momentum tensor depends on ξ and, in general, this is the case for the vacuum energy distribution and hence for the integrated vacuum energy. As a result, as was mentioned in [39], the mode sum energy and integrated energy in general are different. (Note that in most of the papers referred to above the first

quantity is considered.) Below for the geometries under consideration we calculate both these quantities and argue that this difference is due to the existence of an additional surface energy contribution to the total vacuum energy, and the Casimir energy decomposition into volume and surface parts is provided (for a similar consideration in the case of parallel plate geometry see [41]).

We have organized the paper as follows. In the next section we consider the vacuum inside a sphere and derive formulas for expectation values of the energy density and stresses. Section 3 is devoted to the corresponding global quantities, such as the total interior Casimir energy and vacuum force acting on a sphere. We show that the interior vacuum energy contains two parts: volume and surface ones. The vacuum expectation values of the energy-momentum tensor (EMT) for the region outside a sphere are considered in section 4. The expressions for the total energy and force acting on a sphere from outside are derived. The total outside Casimir energy is decomposed into surface and volume parts. Further in this section a spherical shell with zero thickness is considered and the corresponding global quantities including interior and exterior parts are evaluated. In section 5 we consider the vacuum energy-momentum tensor between two concentric spheres and section 6 is devoted to the global quantities in this region. Section 7 concludes the main results of the present paper. In the appendix we derive the summation formula over zeros of the Bessel functions combination on the basis of the generalized Abel-Plana formula. The vacuum expectation values in the region between two spheres contain this type of series.

2 Vacuum EMT inside a spherical shell

Consider a real scalar field φ with curvature coupling parameter ξ in D -dimensional space satisfying the Robin boundary condition

$$(A_1 + B_1 n^i \nabla_i) \varphi(x) = 0 \quad (2.1)$$

on the sphere, where A_1 and B_1 are constants, n^i is the unit inward normal to the sphere, and ∇_i is the covariant derivative operator. Of course, all results in the following will depend on the ratio A_1/B_1 only. However, to keep the transition to the Dirichlet and Neumann cases transparent we will use the form (2.1). The corresponding field equation has the form

$$(\nabla_i \nabla^i + m^2 + \xi R) \varphi = 0, \quad \nabla_i \nabla^i = \frac{1}{\sqrt{-g}} \partial_i \left(\sqrt{-g} g^{ik} \partial_k \right), \quad (2.2)$$

where R is the scalar curvature for the background spacetime. The values $\xi = 0$, and $\xi = \xi_c$ with $\xi_c \equiv (D-1)/4D$ correspond to the minimal and conformal couplings, respectively. Here we will consider the case of flat spacetime. The corresponding metric energy-momentum tensor is defined as (see, e.g., [37])

$$T_{ik} = (1 - 2\xi) \partial_i \varphi \partial_k \varphi + (2\xi - 1/2) g_{ik} \partial^l \varphi \partial_l \varphi - 2\xi \varphi \nabla_i \nabla_k \varphi + (1/2 - 2\xi) m^2 g_{ik} \varphi^2. \quad (2.3)$$

It can be seen that by using the field equation this expression can also be presented in the form

$$T_{ik} = \partial_i \varphi \partial_k \varphi + \left[\left(\xi - \frac{1}{4} \right) g_{ik} \nabla_l \nabla^l - \xi \nabla_i \nabla_k \right] \varphi^2, \quad (2.4)$$

and the corresponding trace is equal to

$$T_i^i = D(\xi - \xi_c) \nabla_i \nabla^i \varphi^2 + m^2 \varphi^2. \quad (2.5)$$

By virtue of Eq.(2.4) for the vacuum expectation values (VEV's) of the EMT we have

$$\langle 0|T_{ik}(x)|0\rangle = \lim_{x' \rightarrow x} \partial_i \partial'_k \langle 0|\varphi(x)\varphi(x')|0\rangle + \left[\left(\xi - \frac{1}{4} \right) g_{ik} \nabla_l \nabla^l - \xi \nabla_i \nabla_k \right] \langle 0|\varphi^2(x)|0\rangle, \quad (2.6)$$

where $|0\rangle$ is the amplitude for the vacuum state. Note that the VEV $\langle 0|\varphi(x)\varphi(x')|0\rangle \equiv G^+(x, x')$ is known as a positive frequency Wightman function. In Eq.(2.6) instead of this function one can choose any other bilinear function of fields such as the Hadamard function, Feynman's Green function, etc. The regularized vacuum EMT does not depend on the specific choice. The expectation values (2.6) are divergent. They are divergent in unbounded Minkowski spacetime as well. In a flat spacetime the regularization is performed by subtracting from Eq.(2.6) the corresponding Minkowskian part:

$$\langle T_{ik}(x) \rangle_{\text{reg}} = \langle 0|T_{ik}(x)|0\rangle - \langle 0_M|T_{ik}(x)|0_M\rangle = \hat{\theta}_{ik} \langle \varphi(x)\varphi(x') \rangle_{\text{reg}}, \quad (2.7)$$

where $|0_M\rangle$ denotes the amplitude for the Minkowski vacuum and the form of the operator $\hat{\theta}_{ik}$ directly follows from Eq.(2.6). Therefore the finite difference between two divergent terms in Eq.(2.7) can be obtained from the corresponding difference between the Wightman functions, by applying a certain second-order differential operator and taking the coincidence limit. To derive the expression for the regularized VEV of the field bilinear product we will use the mode summation method. By expanding the field operator over eigenfunctions and using the commutation rules one can see that

$$\langle 0|\varphi(x)\varphi(x')|0\rangle = \sum_{\alpha} \varphi_{\alpha}(x)\varphi_{\alpha}^*(x'), \quad (2.8)$$

where $\{\varphi_{\alpha}(x), \varphi_{\alpha}^*(x')\}$ is a complete set of positive and negative frequency solutions to the field equation (2.2), satisfying boundary condition (2.1).

In accordance with the symmetry of the problem under consideration we shall use hyperspherical polar coordinates $(r, \vartheta, \phi) \equiv (r, \theta_1, \theta_2, \dots, \theta_n, \phi)$, $n = D - 2$, related to the rectangular ones (x_1, x_2, \dots, x_D) by (see, for instance, [42], Section 11.1)

$$\begin{aligned} x_1 &= r \cos \theta_1, & x_2 &= r \sin \theta_1 \cos \theta_2, \dots, & x_n &= r \sin \theta_1 \sin \theta_2 \dots \sin \theta_{n-1} \cos \theta_n \\ x_{D-1} &= r \sin \theta_1 \sin \theta_2 \dots \sin \theta_n \cos \phi, & x_D &= r \sin \theta_1 \sin \theta_2 \dots \sin \theta_n \sin \phi, \end{aligned} \quad (2.9)$$

where $0 \leq \theta_k \leq \pi$, $k = 1, \dots, n$ and $0 \leq \phi \leq 2\pi$. In the hyperspherical coordinates for the region inside the sphere the complete set of solutions to Eq.(2.2), regular at the origin, has the form

$$\varphi_{\alpha}(x) = \beta_{\alpha} r^{-n/2} J_{\nu}(r \sqrt{\omega^2 - m^2}) Y(m_k; \vartheta, \phi) e^{-i\omega t}, \quad \nu = l + n/2, \quad l = 0, 1, 2, \dots, \quad (2.10)$$

where $m_k = (m_0 \equiv l, m_1, \dots, m_n)$, and m_1, m_2, \dots, m_n are integers such that

$$0 \leq m_{n-1} \leq m_{n-2} \leq \dots \leq m_1 \leq l, \quad -m_{n-1} \leq m_n \leq m_{n-1}, \quad (2.11)$$

$J_{\nu}(z)$ is the Bessel function, and $Y(m_k; \vartheta, \phi)$ is the surface harmonic of degree l (see [42], Section 11.2). This last can be expressed through the Gegenbauer or ultraspherical polynomial $C_p^q(x)$ of degree p and order q as

$$Y(m_k; \vartheta, \phi) = e^{im_n \phi} \prod_{k=1}^n (\sin \theta_k)^{|m_k|} C_{m_{k-1}-|m_k|}^{|m_k|+n/2-k/2}(\sin \theta_k). \quad (2.12)$$

The corresponding normalization integral is in the form

$$\int |Y(m_k; \vartheta, \phi)|^2 d\Omega = N(m_k). \quad (2.13)$$

The explicit form for $N(m_k)$ is given in [42], Section 11.3, and will not be necessary for the following considerations in this paper. From the addition theorem [42], Section 11.4, one has

$$\sum_{m_k} \frac{1}{N(m_k)} Y(m_k; \vartheta, \phi) Y^*(m_k; \vartheta', \phi') = \frac{2l+n}{nS_D} C_l^{n/2}(\cos \theta), \quad (2.14)$$

where $S_D = 2\pi^{D/2}/\Gamma(D/2)$ is the total area of the surface of the unit sphere in D -dimensional space, θ is the angle between directions (ϑ, ϕ) and (ϑ', ϕ') , and sum is taken over the integer values m_k , $k = 1, 2, \dots, n$ in accordance with Eq.(2.11).

The coefficients β_α in Eq.(2.10) can be found from the normalization condition

$$\int |\varphi_\alpha(x)|^2 dV = \frac{1}{2\omega}, \quad (2.15)$$

where the integration goes over the region inside the sphere. Substituting eigenfunctions (2.10), and using the relation (2.13) for the spherical harmonics and the value for the standard integral involving the square of the Bessel function, one finds

$$\beta_\alpha^2 = \frac{\lambda}{N(m_k)\omega a} T_\nu(\lambda a), \quad (2.16)$$

with the notations

$$\lambda = \sqrt{\omega^2 - m^2}, \quad T_\nu(z) = \frac{z}{(z^2 - \nu^2)J_\nu^2(z) + z^2 J_\nu'^2(z)}. \quad (2.17)$$

From boundary condition (2.1) on the sphere surface for eigenfunctions (2.10) one sees that the possible values for the frequency have to be solutions to the following equation

$$AJ_\nu(z) + BzJ_\nu'(z) = 0, \quad z = \lambda a, \quad A = A_1 + B_1 n/2a, \quad B = -B_1/a. \quad (2.18)$$

It is well known (see, e.g., [43, 44]) that for real A , B and $\nu > -1$ all roots of this equation are simple and real, except the case $A/B < -\nu$ when there are two purely imaginary zeros. Let us denote by $\lambda_{\nu,k}$, $k = 1, 2, \dots$, the zeros of the function $AJ_\nu(z) + BzJ_\nu'(z)$ in the right half plane, arranged in ascending order of the real part. Note that for the Neumann boundary condition $A/B = 1 - D/2$, for the TE and TM electromagnetic modes $B = 0$ (Dirichlet) and $A/B = D/2 - 1$, respectively, and for the conformally invariant Hawking boundary condition [40] one has $A/B = 1/D - D/2$.

Substituting Eq.(2.10) into Eq.(2.8) and using addition formula (2.14) for the spherical harmonics, one obtains

$$\begin{aligned} \langle 0 | \varphi(x) \varphi(x') | 0 \rangle &= \frac{(rr')^{-n/2}}{naS_D} \sum_{l=0}^{\infty} (2l+n) C_l^{n/2}(\cos \theta) \\ &\times \sum_{k=1}^{\infty} \frac{\lambda_{\nu,k} T_\nu(\lambda_{\nu,k})}{\sqrt{\lambda_{\nu,k}^2 + m^2 a^2}} J_\nu(\lambda_{\nu,k} r/a) J_\nu(\lambda_{\nu,k} r'/a) e^{i\sqrt{\lambda_{\nu,k}^2/a^2 + m^2}(t'-t)}. \end{aligned} \quad (2.19)$$

To sum over k we will use the generalized Abel-Plana summation formula [31, 32]

$$\begin{aligned} 2 \sum_{k=1}^{\infty} T_\nu(\lambda_{\nu,k}) f(\lambda_{\nu,k}) &= \int_0^\infty f(x) dx + \frac{\pi}{2} \text{Res}_{z=0} f(z) \frac{\bar{Y}_\nu(z)}{\bar{J}_\nu(z)} \\ &- \frac{1}{\pi} \int_0^\infty dx \frac{\bar{K}_\nu(x)}{\bar{I}_\nu(x)} \left[e^{-\nu\pi i} f(xe^{\pi i/2}) + e^{\nu\pi i} f(xe^{-\pi i/2}) \right], \end{aligned} \quad (2.20)$$

where, following [31], for a given function $F(z)$ we use the notation

$$\bar{F}(z) \equiv AF(z) + BzF'(z). \quad (2.21)$$

Formula (2.20) is valid for functions $f(z)$ analytic in the right half plane and satisfying the conditions

$$f(ze^{\pi i}) = -e^{2\nu\pi i}f(z) + o(z^{2\nu-1}), \quad z \rightarrow 0, \quad (2.22)$$

$$|f(z)| < \varepsilon(x)e^{c|y|}, \quad c < 2, \quad \varepsilon(x) \rightarrow 0, \quad x \rightarrow \infty. \quad (2.23)$$

This formula can be generalized in the case of the existence of purely imaginary zeros for the function $\bar{J}_\nu(z)$ by adding the corresponding residue term and taking the principal value of the integral on the right (see [32]). However, in the following we will assume values of A/B for which all these zeros are real.

As the function $f(z)$ in Eq.(2.20) let us choose

$$f(z) = \frac{z}{\sqrt{z^2 + m^2 a^2}} J_\nu(zr/a) J_\nu(zr'/a) e^{i\sqrt{z^2/a^2 + m^2}(t'-t)}. \quad (2.24)$$

This function has branch points on the imaginary axis. As has been shown in [31, 32] the formula (2.20) can be used for functions having this type of branch point as well. As $f(z) \sim z^{2\nu+1}$, $z \rightarrow 0$ condition (2.22) is satisfied. Using the asymptotic formulae for the Bessel function (see, e.g., [44]) it is easy to see that condition (2.23) is satisfied if $r + r' + |t - t'| < 2a$. (Note that this condition is satisfied in the coincidence limit $r = r'$, $t = t'$ for interior points, $r < a$.) Assuming that this is the case and applying to the sum over k in Eq.(2.8) formula (2.20) with $f(z)$ from Eq.(2.24) one obtains

$$\begin{aligned} \langle 0|\varphi(x)\varphi(x')|0\rangle &= \frac{1}{2naS_D} \sum_{l=0}^{\infty} \frac{2l+n}{(rr')^{n/2}} C_l^{n/2}(\cos\theta) \left[\int_0^\infty f(z)dz \right. \\ &\quad \left. - \frac{2}{\pi} \int_{ma}^\infty dz z \frac{\bar{K}_\nu(z)}{\bar{I}_\nu(z)} \frac{I_\nu(zr/a)I_\nu(zr'/a)}{\sqrt{z^2 - m^2 a^2}} \cosh \left[\sqrt{z^2/a^2 - m^2}(t' - t) \right] \right], \end{aligned} \quad (2.25)$$

where we have used the result that the difference of the radicals is nonzero above the branch point only and introduced the modified Bessel functions. The contribution of the term in the first integral to the VEV does not depend on a , whereas the contribution of the second one tends to zero as $a \rightarrow \infty$. It follows from here that the first term is the corresponding function for the unbounded Minkowski space:

$$\langle 0_M|\varphi(x)\varphi(x')|0_M\rangle = \frac{1}{2nS_D} \sum_{l=0}^{\infty} \frac{2l+n}{(rr')^{n/2}} C_l^{n/2}(\cos\theta) \int_0^\infty dz \frac{ze^{i\sqrt{z^2+m^2}(t'-t)}}{\sqrt{z^2+m^2}} J_\nu(zr) J_\nu(zr'). \quad (2.26)$$

This can be seen also by direct evaluation. Indeed, the sum over l can be summarized using the Gegenbauer addition theorem for the Bessel function [44]. The value of the remaining integral involving the Bessel function can be found, e.g., in [45]. As a result we obtain the standard expression for the D -dimensional Minkowskian Wightman function:

$$\langle 0_M|\varphi(x)\varphi(x')|0_M\rangle = \frac{m^{(D-1)/2}}{(2\pi)^{(D+1)/2}} \frac{K_{(D-1)/2} \left(m\sqrt{(\mathbf{x} - \mathbf{x}')^2 - (t - t')^2 + i\varepsilon} \right)}{[(\mathbf{x} - \mathbf{x}')^2 - (t - t')^2 + i\varepsilon]^{(D-1)/4}}, \quad (2.27)$$

where $x = (t, \mathbf{x})$, $\varepsilon > 0$ for $t > t'$, and $\varepsilon < 0$ for $t < t'$. Using the Feynman Green function in Eq.(2.6) we would obtain the same expression (2.27) with $\varepsilon > 0$.

Hence we see that the application of the generalized Abel-Plana formula allows us to extract from the bilinear field product the contribution due to the unbounded Minkowski spacetime. Combining Eqs.(2.19) and (2.26) for the regularized Wightmann function one obtains

$$\begin{aligned} \langle \varphi(x)\varphi(x') \rangle_{\text{reg}} &= \langle 0|\varphi(x)\varphi(x')|0 \rangle - \langle 0_M|\varphi(x)\varphi(x')|0_M \rangle = \frac{-1}{n\pi S_D} \sum_{l=0}^{\infty} \frac{2l+n}{(rr')^{n/2}} C_l^{n/2}(\cos \theta) \\ &\times \int_m^{\infty} dz z \frac{\bar{K}_\nu(za)}{\bar{I}_\nu(za)} \frac{I_\nu(zr)I_\nu(zr')}{\sqrt{z^2 - m^2}} \cosh \left[\sqrt{z^2 - m^2}(t' - t) \right]. \end{aligned} \quad (2.28)$$

Let us recall that the response of the particle detector at a given state of motion is determined by this function [37, 46]. Performing the limit $x' \rightarrow x$ and using the value

$$C_l^{n/2}(1) = \frac{\Gamma(l+n)}{\Gamma(n)l!} \quad (2.29)$$

(see [42], Section 11.1) for the regularized field square we get

$$\langle \varphi^2(r) \rangle_{\text{reg}} = -\frac{1}{\pi a r^n S_D} \sum_{l=0}^{\infty} D_l \int_{ma}^{\infty} dz \frac{\bar{K}_\nu(z)}{\bar{I}_\nu(z)} \frac{z I_\nu^2(zr/a)}{\sqrt{z^2 - m^2 a^2}}, \quad (2.30)$$

where

$$D_l = (2l + D - 2) \frac{\Gamma(l + D - 2)}{\Gamma(D - 1) l!} \quad (2.31)$$

is the degeneracy of each angular mode with given l .

The VEV for the EMT can be evaluated by substituting Eqs.(2.28) and (2.30) into Eq.(2.7), where the operator $\hat{\theta}_{ik}$ is defined in accordance with Eq.(2.6). From the symmetry of the problem under consideration it follows that $\langle T_{ik} \rangle_{\text{reg}}$ is a combination of the second rank tensors constructed from the metric g_{ik} , the unit vector \hat{t}^i in the time direction, and the unit vector n^i in the radial direction. We will present this combination in the form

$$\langle T_i^k \rangle_{\text{reg}} = \varepsilon \hat{t}_i \hat{t}^k + p n_i n^k + p_\perp \left(\hat{t}_i \hat{t}^k - n_i n^k - \delta_i^k \right) = \text{diag}(\varepsilon, -p, -p_\perp, \dots, -p_\perp), \quad (2.32)$$

where the vacuum energy density ε and the effective pressures in radial, p , and azimuthal, p_\perp , directions are functions of the radial coordinate only. As a consequence of the continuity equation $\nabla_k \langle T_i^k \rangle_{\text{reg}} = 0$, these functions are related by the equation

$$r \frac{dp}{dr} + (D - 1)(p - p_\perp) = 0. \quad (2.33)$$

From Eqs.(2.7), (2.28), and (2.30) for the EMT components one obtains

$$q(a, r) = -\frac{1}{2\pi a^3 r^n S_D} \sum_{l=0}^{\infty} D_l \int_{ma}^{\infty} dz z^3 \frac{\bar{K}_\nu(z)}{\bar{I}_\nu(z)} \frac{F_\nu^{(q)}[I_\nu(zr/a)]}{\sqrt{z^2 - m^2 a^2}}, \quad q = \varepsilon, p, p_\perp, \quad r < a, \quad (2.34)$$

where for a given function $f(y)$ we have introduced the notations

$$F_\nu^{(\varepsilon)}[f(y)] = (1 - 4\xi) \left[f'^2(y) - \frac{n}{y} f(y) f'(y) + \left(\frac{\nu^2}{y^2} - \frac{1 + 4\xi - 2(mr/y)^2}{1 - 4\xi} \right) f^2(y) \right] \quad (2.35)$$

$$F_\nu^{(p)}[f(y)] = f'^2(y) + \frac{\xi_1}{y} f(y) f'(y) - \left(1 + \frac{\nu^2 + \xi_1 \frac{n}{2}}{y^2} \right) f^2(y), \quad \xi_1 = 4(n + 1)\xi - n \quad (2.36)$$

$$F_\nu^{(p_\perp)}[f(y)] = (4\xi - 1) f'^2(y) - \frac{\xi_1}{y} f(y) f'_\nu(y) + \left[4\xi - 1 + \frac{\nu^2(1 + \xi_1) + \xi_1 \frac{n}{2}}{(n + 1)y^2} \right] f^2(y). \quad (2.37)$$

It can easily be seen that components (2.34) satisfy Eq.(2.33) and are finite for $r < a$. The formulas (2.34) may be derived in another, equivalent way, introducing into the divergent mode sum

$$\langle 0|T_{ik}\{\varphi(x), \varphi(x)\}|0\rangle = \sum_{\alpha} T_{ik}\{\varphi_{\alpha}(x), \varphi_{\alpha}^*(x)\} \quad (2.38)$$

a cutoff function and applying the summation formula (2.20). The latter allows one to extract the corresponding Minkowskian part by a manifestly cutoff independent method.

At the sphere center the nonzero contribution to VEV (2.34) comes from the summands with $l = 0$ and $l = 1$ and one has

$$\begin{aligned} \varepsilon(a, 0) &= \frac{1}{2^D \pi^{D/2+1} \Gamma(D/2)} \int_m^{\infty} \frac{z^{D+1} dz}{\sqrt{z^2 - m^2}} \\ &\times \left[\left(4\xi + 1 - 2\frac{m^2}{z^2} \right) \frac{\bar{K}_{D/2-1}(az)}{\bar{I}_{D/2-1}(az)} + (4\xi - 1) \frac{\bar{K}_{D/2}(az)}{\bar{I}_{D/2}(az)} \right], \end{aligned} \quad (2.39)$$

$$\begin{aligned} p(a, 0) &= p_{\perp}(a, 0) = -\frac{1}{2^D \pi^{D/2+1} D \Gamma(D/2)} \int_m^{\infty} \frac{z^{D+1} dz}{\sqrt{z^2 - m^2}} \\ &\times \left[(\xi_1 - 2) \frac{\bar{K}_{D/2-1}(az)}{\bar{I}_{D/2-1}(az)} + \xi_1 \frac{\bar{K}_{D/2}(az)}{\bar{I}_{D/2}(az)} \right]. \end{aligned} \quad (2.40)$$

Note that for the conformally coupled massless scalar

$$\varepsilon(a, 0) = Dp(a, 0). \quad (2.41)$$

This can also be obtained directly from the zero trace condition. The results of the corresponding numerical evaluation for the Dirichlet and Neumann minimally and conformally coupled scalars in $D = 3$ are presented in Fig. 1. In the limit of large mass, $ma \gg 1$, using the asymptotic

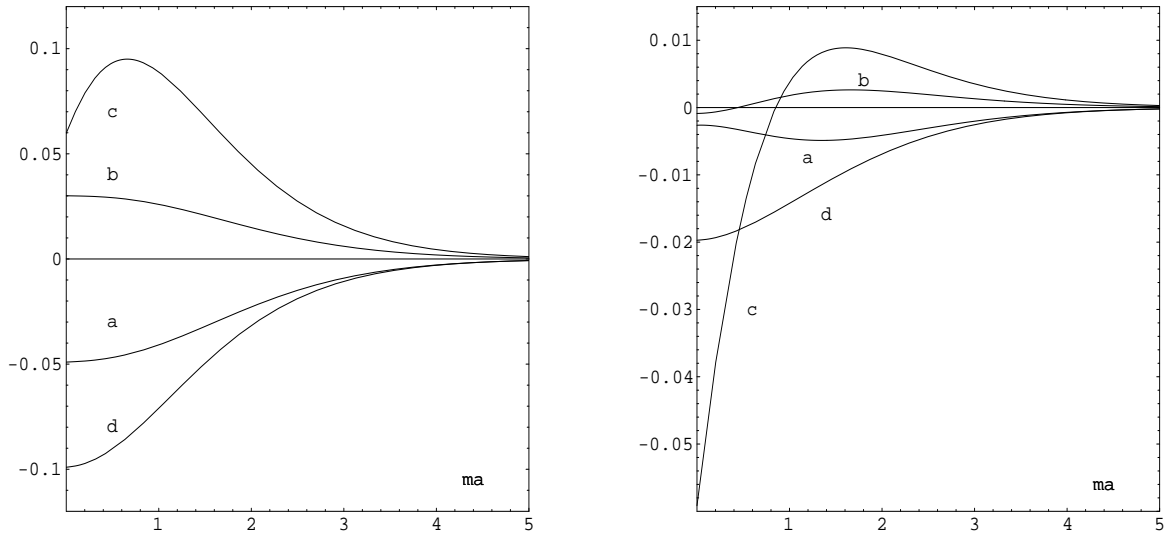


Figure 1: The Casimir energy density, $a^{D+1}\varepsilon$, (a, c) and vacuum pressure, $a^{D+1}p$, (b, d) at sphere center for minimally (left) and conformally (right) coupled Dirichlet (a, b) and Neumann (c, d) scalars in $D = 3$ versus ma .

formulas for the modified Bessel functions for large arguments and the value of the integral

$$\int_{\mu}^{\infty} \frac{z^{m+1} e^{-2z}}{\sqrt{z^2 - \mu^2}} dz = (-1)^m \mu^{m+1} \frac{\partial^m K_1(2\mu)}{\partial(2\mu)^m}, \quad (2.42)$$

from Eqs.(2.39) and (2.40) one obtains

$$\varepsilon(a, 0) \approx -\frac{D}{D-1} p(a, 0) \approx \frac{(4\xi - 1)(am)^{D+1/2} e^{-2am}}{2^D \pi^{(D-1)/2} \Gamma(D/2) a^{D+1}} (2\delta_{B0} - 1). \quad (2.43)$$

The expectation values (2.34) diverge at the sphere surface, $r \rightarrow a$ (note that for a given l the integrals over z diverge as $(a - r)^{-2}$). The corresponding asymptotic behavior can be found using the uniform asymptotic expansions for the modified Bessel functions, and the leading terms have the form

$$p \sim \frac{(D-1)\Gamma((D+1)/2)(\xi - \xi_c)}{2^D \pi^{(D+1)/2} a (a-r)^D} (1 - 2\delta_{B0}), \quad (2.44)$$

$$\varepsilon \sim -p_{\perp} \sim -\frac{D\Gamma((D+1)/2)(\xi - \xi_c)}{2^D \pi^{(D+1)/2} (a-r)^{D+1}} (1 - 2\delta_{B0}). \quad (2.45)$$

These terms do not depend on mass or Robin coefficient B_1 , and have opposite signs for Dirichlet and Neumann boundary conditions. Surface divergences in renormalized expectation values for EMT are well known in quantum field theory with boundaries [6, 39, 40]. In the case of $D = 3$ massless fields the corresponding asymptotic series near an arbitrary smooth boundary are presented in [39, 40]. In particular, for $D = 3$, $\xi = 0$ from Eqs.(2.44) and (2.45) we obtain the leading terms given in [39] for the minimally coupled Dirichlet and Neumann scalar fields. Taking the limit $a, r \rightarrow \infty$, $a - r = \text{const}$, one obtains the leading terms for the asymptotic behavior near the single plate (see, for instance, [41]). For a conformally coupled scalar the coefficients for the leading terms are zero and $\varepsilon, p_{\perp} \sim (a - r)^{-D}$, $p \sim (a - r)^{1-D}$. In general asymptotic series can be developed in powers of the distance from the boundary. The corresponding subleading coefficients will depend on the mass, Robin coefficient, and sphere radius.

In the case $D = 1$ we obtain VEV's for the one - dimensional segment $-a \leq x \leq a$. Due to the gamma function in the denominator of the expression for D_l in Eq.(2.31), now the only nonzero coefficients are $D_0 = D_1 = 1$. Using the standard expressions for the functions $I_{\pm 1/2}(z)$ and $K_{\pm 1/2}(z)$ we can easily see that $F_{\pm 1/2}^{(p)}[I_{\pm 1/2}(z)] = \pm 2/(\pi z)$, and hence the vacuum stresses are uniform:

$$p = -\frac{1}{\pi} \int_m^{\infty} \frac{z^2 dz}{\sqrt{z^2 - m^2}} \frac{1}{e^{4az} \left(\frac{B_1 z - A_1}{B_1 z + A_1} \right)^2 - 1}. \quad (2.46)$$

For $B_1 = 0$ and $A_1 = 0$ we obtain from here the standard results for Dirichlet and Neumann scalars. Unlike the stress distribution the energy distribution is nonuniform. The corresponding expression directly follows from Eq.(2.34):

$$\varepsilon = -\frac{1}{\pi} \int_m^{\infty} dz \frac{\sqrt{z^2 - m^2}}{e^{4az} \left(\frac{B_1 z - A_1}{B_1 z + A_1} \right)^2 - 1} \left[1 + \left(1 + \frac{4\xi - 1}{1 - m^2/z^2} \right) \frac{B_1 z - A_1}{B_1 z + A_1} e^{2az} \cosh 2zr \right]. \quad (2.47)$$

Note that for the case of massless field the expressions (2.46) and (2.47) coincide with the general formulas derived in [41] for two - plate geometry if one takes there $\beta_1 = \beta_2 = B_1/A_1$ and $D = 1$. In $D = 1$ our assumption that all zeros for $\bar{J}_{\nu}(z)$ are real corresponds to $B_1/A_1 < 0$.

3 Total Casimir energy and vacuum forces inside a sphere

In the previous section we considered the scalar vacuum densities and stresses inside a sphere. Here we will concentrate on the corresponding global quantities. First of all note that by using formulae (2.19) and (2.6) the unregularized 00-component for the vacuum EMT may be presented in the form

$$\langle 0|T_{00}(x)|0\rangle = \frac{1}{2a^3 S_D r^n} \sum_{l=0}^{\infty} D_l \sum_{k=1}^{\infty} \frac{\lambda_{\nu,k}^3 T_{\nu}(\lambda_{\nu,k})}{\sqrt{\lambda_{\nu,k}^2 + m^2 a^2}} f_{\nu}^{(0)}[J_{\nu}(\lambda_{\nu,k} r/a)], \quad (3.1)$$

where we have introduced the notation

$$f_{\nu}^{(0)}[f(y)] = (1 - 4\xi) \left[f'^2(y) - \frac{n}{y} f(y) f'(y) + \left(\frac{\nu^2}{y^2} - \frac{4\xi + 1}{4\xi - 1} \right) f^2(y) \right] + 2 \left(\frac{mr}{y} \right)^2 f^2(y). \quad (3.2)$$

By using the standard integrals involving the Bessel functions for the total volume energy inside a spherical shell one finds

$$E_{\text{in}}^{(\text{vol})} = \frac{1}{2a} \sum_{l=0}^{\infty} D_l \sum_{k=1}^{\infty} \sqrt{\lambda_{\nu,k}^2 + m^2 a^2} \left[1 + \frac{(4\xi - 1) A_1 B \lambda_{\nu,k}^2}{(\lambda_{\nu,k}^2 + m^2 a^2) [A^2 + B^2 (\lambda_{\nu,k}^2 - \nu^2)]} \right]. \quad (3.3)$$

As we see in the general Robin case this energy differs from the total vacuum energy inside a sphere:

$$E_{\text{in}} = \frac{1}{2} \sum_{l=0}^{\infty} D_l \sum_{k=1}^{\infty} \omega_{\nu,k}, \quad \omega_{\nu,k} = \sqrt{\lambda_{\nu,k}^2/a^2 + m^2}. \quad (3.4)$$

We will interpret this difference as a result of the presence of an additional surface energy contribution:

$$E_{\text{in}}^{(\text{surf})} = -\frac{(4\xi - 1) A_1 B}{2a} \sum_{l=0}^{\infty} D_l \sum_{k=1}^{\infty} \frac{\lambda_{\nu,k}^2}{\sqrt{\lambda_{\nu,k}^2 + m^2 a^2} [A^2 + B^2 (\lambda_{\nu,k}^2 - \nu^2)]}, \quad (3.5)$$

such that

$$E_{\text{in}} = E_{\text{in}}^{(\text{vol})} + E_{\text{in}}^{(\text{surf})}. \quad (3.6)$$

The surface energy (3.5) can be obtained independently by integrating the corresponding surface energy density. To see this let us note that there is a surface energy density contribution to the total energy density in the form [40]

$$T_{00}^{(\text{surf})} = -(2\xi - 1/2) \delta(x, \partial M) \varphi n^i \partial_i \varphi, \quad (3.7)$$

where $\delta(x, \partial M)$ - is a "one sided" δ -distribution. By using Eq.(2.19) the corresponding VEV takes the form

$$\begin{aligned} \langle 0|T_{00}^{(\text{surf})}(x)|0\rangle &= (2\xi - 1/2) \delta(x, \partial M) \left(\partial_r \langle 0|\varphi(r) \varphi(r')|0\rangle \right)_{r'=r} = -(4\xi - 1) \frac{\delta(r - a + 0)}{2a S_D r^{D-1}} \\ &\times \sum_{l=0}^{\infty} D_l \sum_{k=1}^{\infty} \frac{\lambda_{\nu,k} T_{\nu}(\lambda_{\nu,k})}{\sqrt{\lambda_{\nu,k}^2 + m^2 a^2}} J_{\nu}(z) \left[\frac{n}{2} J_{\nu}(z) - z J'_{\nu}(z) \right], \quad z = \lambda_{\nu,k} r/a \end{aligned} \quad (3.8)$$

Integrating this density over the region inside a sphere, we obtain the corresponding surface energy:

$$E_{\text{in}}^{(\text{surf})} = \int d^D x \langle 0|T_{00}^{(\text{surf})}(x)|0\rangle. \quad (3.9)$$

It can easily be seen that this expression with Eq.(3.8) coincides with Eq.(3.5).

The subtracted surface energy density can be obtained by applying to the sum over k in Eq.(3.8) the summation formula (2.20) and omitting the term coming from the first integral on the right of this formula. An alternative way is to use in Eq.(3.8) the subtracted product $\langle \varphi(r)\varphi(r') \rangle_{SUB}$ from Eq.(2.28) instead of $\langle 0|\varphi(r)\varphi(r')|0 \rangle$. As a result one obtains

$$\left\langle T_{00}^{(\text{surf})}(x) \right\rangle_{SUB} = \delta(r - a + 0) \frac{1 - 4\xi}{2\pi a S_D r^{D-1}} \sum_{l=0}^{\infty} D_l \int_{ma}^{\infty} \frac{z dz}{\sqrt{z^2 - m^2 a^2}} \frac{\bar{K}_\nu(z)}{\bar{I}_\nu(z)} F_s[I_\nu(zr/a)], \quad (3.10)$$

where for a given function $f(y)$ we have introduced the notation

$$F_s[f(y)] = f(y) \left[y f'(y) - \frac{n}{2} f(y) \right]. \quad (3.11)$$

Integration of this energy density gives the total subtracted surface energy:

$$E_{\text{in}}^{(\text{surf})} = \frac{1 - 4\xi}{2\pi a} \sum_{l=0}^{\infty} D_l \int_{ma}^{\infty} \frac{z dz}{\sqrt{z^2 - m^2 a^2}} \frac{\bar{K}_\nu(z)}{\bar{I}_\nu(z)} F_s[I_\nu(z)]. \quad (3.12)$$

Integrating the energy density $q = \varepsilon$ in Eq.(2.34) over the volume inside a sphere we obtain the corresponding subtracted volume energy. Using the result that for any modified cylinder function $Z_\nu(y) = c_1 I_\nu(y) + c_2 K_\nu(y)$ we have the formula

$$\int dr r F_\nu^{(\varepsilon)}[Z_\nu(zr)] = \frac{1}{z^2} F_v[Z_\nu(zr)] \quad (3.13)$$

with

$$F_v[f(y)] = (1 - 4\xi) F_s[f(y)] + (y^2 - m^2 a^2) \left[f'^2(y) - \left(1 + \frac{\nu^2}{y^2} \right) f^2(y) \right], \quad (3.14)$$

one obtains

$$E_{\text{in}}^{(\text{vol})} = -\frac{1}{2\pi a} \sum_{l=0}^{\infty} D_l \int_{ma}^{\infty} \frac{\bar{K}_\nu(z)}{\bar{I}_\nu(z)} \frac{z dz}{\sqrt{z^2 - m^2 a^2}} F_v[I_\nu(z)]. \quad (3.15)$$

Now, using relations (3.12) and (3.15) for the total Casimir energy inside a sphere, we get

$$E_{\text{in}} = -\frac{1}{2\pi a} \sum_{l=0}^{\infty} D_l \int_{ma}^{\infty} \frac{dz}{z} \sqrt{z^2 - m^2 a^2} \frac{\bar{K}_\nu(z)}{\bar{I}_\nu(z)} F_t[I_\nu(z)], \quad (3.16)$$

where for future convenience we have defined

$$F_t[f(y)] = y^2 f'^2(y) - (y^2 + \nu^2) f^2(y). \quad (3.17)$$

As we see the dependences on the coupling parameter ξ in volume and surface energies canceled each other and the total energy does not depend on this parameter. We might expect this result as the eigenmodes do not depend on ξ . With the help of the recurrence relations for $I_\nu(z)$ it can be seen that for the case $D = 3$ the Dirichlet massless scalar field interior energy (3.16) coincides with the result given in [38]. The force acting per unit area of the sphere from inside is determined as

$$F_{\text{in}} = p|_{r=a-} = \frac{-1}{2\pi a^{D+1} S_D} \sum_{l=0}^{\infty} D_l \int_{ma}^{\infty} \frac{z^3 dz}{\sqrt{z^2 - m^2 a^2}} \frac{\bar{K}_\nu(z)}{\bar{I}_\nu(z)} F_\nu^{(p)}[I_\nu(z)], \quad (3.18)$$

with the notation (2.36). Note that one has the relation

$$z^2 F_\nu^{(p)} = F_t[f(z)] + \xi_1 F_s[f(z)]. \quad (3.19)$$

For the massless scalar this leads to the formula

$$a^D S_D F_{\text{in}} = E_{\text{in}}^{(\text{vol})} + \frac{4D}{4\xi - 1} (\xi - \xi_c) E_{\text{in}}^{(\text{surf})}. \quad (3.20)$$

This relation can also be derived directly from continuity equation (2.33), taking into account the formula

$$(D - 1)p_{\perp} = \varepsilon - p - D(\xi - \xi_c) \nabla_i \nabla^i \langle \varphi^2 \rangle_{\text{reg}}, \quad (3.21)$$

where we have used expression (2.5) for the EMT trace in the case of a massless field. Substituting Eq.(3.21) into Eq.(2.33) and integrating over the interior region we obtain Eq.(3.20). Note that for the minimally coupled scalar ($\xi = 0$) this formula may be presented in the form of the standard thermodynamic relation

$$dE_{\text{in}}^{(\text{vol})} = -F_{\text{in}} dv_D + \sigma_{\text{in}} ds_D, \quad (3.22)$$

where $v_D = S_D a^D / D$ and $s_D = S_D a^{D-1}$ are the volume and surface area for a D -dimensional ball with radius a , and $\sigma_{\text{in}} = E_{\text{in}}^{(\text{surf})} / s_D$ is the inner surface tension.

The expressions (3.12), (3.15), (3.16) and (3.18) are divergent in the given forms. Moreover, in the sums over l the individual integrals diverge. To extract finite results we can apply the procedure already used in [9, 10]. A detailed description of this and the results for the corresponding numerical evaluations will be reported elsewhere. Here we just briefly outline the scheme with the example of the surface energy (3.12).

First of all we note that [9]

$$\sum_{l=0}^{\infty} D_l = 0, \quad D < 1. \quad (3.23)$$

As a result for these values D we can add to the integrand any l -independent term without changing the value of the sum. We choose this term in a way that makes the z -integral finite. For this we write the asymptotic behavior of the integrand in Eq.(3.12) as $z \rightarrow \infty$:

$$\frac{\bar{K}_{\nu}(z)}{\bar{I}_{\nu}(z)} F_s[I_{\nu}(z)] \sim \left(\delta_{B0} - \frac{1}{2} \right) \left(1 - \frac{S^{(as)}}{2z} + \dots \right), \quad (3.24)$$

where we introduced the notation

$$S^{(as)} = \begin{cases} D - 3 + 4A/B, & B \neq 0 \\ D - 1, & B = 0. \end{cases} \quad (3.25)$$

Note that the displayed terms are independent of l . Hence, without changing the value of the surface energy for $D < 1$ we can write expression (3.12) as

$$E_{\text{in}}^{(\text{surf})} = \frac{1 - 4\zeta}{2\pi a} \sum_{l=0}^{\infty} D_l \int_{ma}^{\infty} \frac{z dz}{\sqrt{z^2 - m^2 a^2}} \left\{ \frac{\bar{K}_{\nu}(z)}{\bar{I}_{\nu}(z)} F_s[I_{\nu}(z)] - \left(\delta_{B0} - \frac{1}{2} \right) \left(1 - \frac{S^{(as)}}{2z} \right) \right\}. \quad (3.26)$$

In this form the integrals are well defined for a fixed l , as for large $z \rightarrow \infty$ the subintegrand behaves as z^{-2} . Then we analytically continue expression (3.26) to all D . To turn the sum over l into a convergent series we can use procedure often used in the previous literature on the Casimir effect (see, for instance, [11, 12, 9, 10, 13]). First we rescale the integration variable, $z \rightarrow z\nu$. Then we add and subtract the leading terms in the asymptotic expansion of the subintegrand for $l \rightarrow \infty$ using the uniform asymptotic expansions for the modified Bessel functions. The subtracted part is finite and may be numerically evaluated, and the asymptotic part can be expressed via Riemann or Hurwitz zeta functions. Note that in the final result divergent terms

in the form of poles for the latter functions may remain. Here one should take into account that the local cutoff dependent parts are automatically lost in this regularization procedure. The corresponding terms may be important in comparing the results with experiments (for instance, in the electromagnetic case, see, e.g., [47, 48]), and can be extracted by expanding the local expectation values (2.34) in powers of the distance from the boundary.

4 Vacuum outside a sphere

To obtain the VEV for the EMT outside a sphere we consider first the scalar vacuum in the layer between two concentric spheres with radii a and b , $a < b$. The corresponding boundary conditions have the form

$$\left(\tilde{A}_r + \tilde{B}_r \frac{\partial}{\partial r} \right) \varphi(x) = 0, \quad r = a, b, \quad (4.1)$$

with constant coefficients \tilde{A}_r and \tilde{B}_r , in general, different for the inner and outer spheres. Now the complete set of solutions to the field equation has the form (2.10) with the replacement

$$J_\nu(\lambda r) \rightarrow g_\nu(\lambda a, \lambda r) \equiv J_\nu(\lambda r) \bar{Y}_\nu^{(a)}(\lambda a) - \bar{J}_\nu^{(a)}(\lambda a) Y_\nu(\lambda r), \quad \lambda = \sqrt{\omega^2 - m^2}, \quad (4.2)$$

where $Y_\nu(z)$ is the Neumann function, and functions with overbars are defined in analogy to Eq.(2.21):

$$\bar{F}^{(\alpha)}(z) \equiv A_\alpha F(z) + B_\alpha z F'(z), \quad A_\alpha = \tilde{A}_\alpha - \tilde{B}_\alpha n / 2\alpha, \quad B_\alpha = \tilde{B}_\alpha / \alpha, \quad \alpha = a, b. \quad (4.3)$$

The eigenfunctions (4.2) satisfy the boundary condition on the sphere $r = a$. From the boundary condition on $r = b$ one obtains that the corresponding eigenmodes are solutions to the equation

$$C_\nu^{ab}(b/a, \lambda a) \equiv \bar{J}_\nu^{(a)}(\lambda a) \bar{Y}_\nu^{(b)}(\lambda b) - \bar{J}_\nu^{(b)}(\lambda b) \bar{Y}_\nu^{(a)}(\lambda a) = 0. \quad (4.4)$$

The coefficients β_α are determined from the normalization condition (2.16), where now the integration goes over the region between the spheres, $a \leq r \leq b$. Using the formula for integrals involving the product of any two cylinder functions one obtains

$$\beta_\alpha^2 = \frac{\pi^2 \lambda}{4N(m_k) \omega a} T_\nu^{ab}(b/a, \lambda a), \quad (4.5)$$

where $N(m_k)$ comes from the normalization integral (2.13) and we use notation (A.6).

Substituting the eigenfunctions into the mode sum (2.8) and using addition formula (2.14) for the expectation value of the field product one finds

$$\langle 0 | \varphi(x) \varphi(x') | 0 \rangle = \frac{\pi^2 (rr')^{-n/2}}{4naS_D} \sum_{l=0}^{\infty} (2l+n) C_l^{n/2}(\cos \theta) \sum_{k=1}^{\infty} h(\gamma_{\nu,k}) T_\nu^{ab}(b/a, \gamma_{\nu,k}), \quad (4.6)$$

with $\gamma_{\nu,k} = \lambda a$ being the solutions to Eq.(4.4) (see also the Appendix) and

$$h(z) = \frac{z}{\sqrt{z^2 + m^2 a^2}} g_\nu(z, zr/a) g_\nu(z, zr'/a) e^{i\sqrt{z^2/a^2 + m^2}(t'-t)}. \quad (4.7)$$

To sum over k we will use the summation formula (A.11). The corresponding conditions are satisfied if $r + r' + |t - t'| < 2b$. Note that this is the case in the coincidence limit for the region under consideration. Applying to the sum over k in Eq.(4.6) formula (A.11) one obtains

$$\langle 0 | \varphi(x) \varphi(x') | 0 \rangle = \frac{1}{2naS_D} \sum_{l=0}^{\infty} \frac{2l+n}{(rr')^{n/2}} C_l^{n/2}(\cos \theta) \left\{ \int_0^\infty \frac{h(z) dz}{\bar{J}_\nu^{(a)2}(z) + \bar{Y}_\nu^{(a)2}(z)} \right\} \quad (4.8)$$

$$- \frac{2}{\pi} \int_{ma}^{\infty} \frac{z dz}{\sqrt{z^2 - a^2 m^2}} \frac{\bar{K}_{\nu}^{(b)}(\eta z)}{\bar{K}_{\nu}^{(a)}(z)} \frac{G_{\nu}^{(a)}(z, zr/a) G_{\nu}^{(a)}(z, zr'/a)}{\bar{K}_{\nu}^{(a)}(z) \bar{I}_{\nu}^{(b)}(\eta z) - \bar{K}_{\nu}^{(b)}(\eta z) \bar{I}_{\nu}^{(a)}(z)} \cosh \left[\sqrt{z^2/a^2 - m^2} (t' - t) \right] \Bigg\},$$

where we have introduced notations

$$G_{\nu}^{(\alpha)}(z, y) = I_{\nu}(y) \bar{K}_{\nu}^{(\alpha)}(z) - \bar{I}_{\nu}^{(\alpha)}(z) K_{\nu}(y), \quad \alpha = a, b \quad (4.9)$$

(the function with $\alpha = b$ will be used below) with the modified Bessel functions. Note that we have assumed values A_{α} and B_{α} for which all zeros for Eq.(4.4) are real and have omitted the residue terms (A.14). In the following we will consider this case only.

To obtain the vacuum EMT components outside a single sphere let us consider the limit $b \rightarrow \infty$. In this limit the second integral on the right of Eq.(4.8) tends to zero (for large b/a the subintegrand is proportional to $e^{-2bz/a}$), whereas the first one does not depend on b . It follows from here that the quantity

$$\begin{aligned} \langle 0 | \varphi(x) \varphi(x') | 0 \rangle &= \frac{1}{2naS_D} \sum_{l=0}^{\infty} \frac{2l+n}{(rr')^{n/2}} C_l^{n/2}(\cos \theta) \\ &\times \int_0^{\infty} \frac{z dz}{\sqrt{z^2 + m^2 a^2}} \frac{g_{\nu}(z, zr/a) g_{\nu}(z, zr'/a)}{\bar{J}_{\nu}^{(a)2}(z) + \bar{Y}_{\nu}^{(a)2}(z)} e^{i\sqrt{z^2/a^2 + m^2}(t' - t)} \end{aligned} \quad (4.10)$$

is the Wightman function for the exterior region of a single sphere with radius a . To regularize this expression we have to subtract the corresponding part for the unbounded space, which, as we saw, can be presented in the form (2.26). Using the relation

$$\frac{g_{\nu}(z, zr/a) g_{\nu}(z, zr'/a)}{\bar{J}_{\nu}^{(a)2}(z) + \bar{Y}_{\nu}^{(a)2}(z)} - J_{\nu}(zr/a) J_{\nu}(zr'/a) = -\frac{1}{2} \sum_{\sigma=1}^2 \frac{\bar{J}_{\nu}^{(a)}(z)}{\bar{H}_{\nu}^{(\sigma a)}(z)} H_{\nu}^{(\sigma)}(zr/a) H_{\nu}^{(\sigma)}(zr'/a) \quad (4.11)$$

with $H_{\nu}^{(\sigma)}(z)$, $\sigma = 1, 2$, being the Hankel functions, one obtains

$$\begin{aligned} \langle 0 | \varphi(x) \varphi(x') | 0 \rangle - \langle 0_M | \varphi(x) \varphi(x') | 0_M \rangle &= -\frac{1}{4nS_D} \sum_{l=0}^{\infty} \frac{2l+n}{(rr')^{n/2}} C_l^{n/2}(\cos \theta) \\ &\times \sum_{\sigma=1}^2 \int_0^{\infty} dz z \frac{e^{i\sqrt{z^2 + m^2}(t' - t)}}{\sqrt{z^2 + m^2}} \frac{\bar{J}_{\nu}^{(a)}(za)}{\bar{H}_{\nu}^{(\sigma a)}(za)} H_{\nu}^{(\sigma)}(zr) H_{\nu}^{(\sigma)}(zr'). \end{aligned} \quad (4.12)$$

Assuming that the function $\bar{H}_{\nu}^{(1a)}(z)$, $(\bar{H}_{\nu}^{(2a)}(z))$ has no zeros for $0 < \arg z \leq \pi/2$ ($-\pi/2 \leq \arg z < 0$) we can rotate the integration contour on the right by the angle $\pi/2$ for $\sigma = 1$ and by the angle $-\pi/2$ for $\sigma = 2$. The integrals over $(0, ima)$ and $(0, -ima)$ cancel out and after introducing the Bessel modified functions one obtains

$$\begin{aligned} \langle \varphi(x) \varphi(x') \rangle_{\text{reg}} &= -\frac{1}{n\pi S_D} \sum_{l=0}^{\infty} \frac{2l+n}{(rr')^{n/2}} C_l^{n/2}(\cos \theta) \\ &\times \int_m^{\infty} dz z \frac{\bar{I}_{\nu}^{(a)}(za)}{\bar{K}_{\nu}^{(a)}(za)} \frac{K_{\nu}(zr) K_{\nu}(zr')}{\sqrt{z^2 - m^2}} \cosh \left[\sqrt{z^2 - m^2} (t' - t) \right]. \end{aligned} \quad (4.13)$$

For the VEV of the field square in the outside region this leads to

$$\langle \varphi^2(r) \rangle_{\text{reg}} = -\frac{1}{\pi a r^n S_D} \sum_{l=0}^{\infty} D_l \int_{ma}^{\infty} dz z \frac{\bar{I}_{\nu}^{(a)}(z)}{\bar{K}_{\nu}^{(a)}(z)} \frac{K_{\nu}^2(zr/a)}{\sqrt{z^2 - m^2 a^2}}, \quad (4.14)$$

where D_l is defined in accord with Eq.(2.31).

As in the interior case the vacuum EMT is diagonal and the corresponding components can be presented in the form

$$q(a, r) = -\frac{1}{2\pi a^3 r^n S_D} \sum_{l=0}^{\infty} D_l \int_{ma}^{\infty} dz z^3 \frac{\bar{I}_{\nu}^{(a)}(z)}{\bar{K}_{\nu}^{(a)}(z)} \frac{F_{\nu}^{(q)}[K_{\nu}(zr/a)]}{\sqrt{z^2 - m^2 a^2}}, \quad q = \varepsilon, p, p_{\perp}, \quad r > a, \quad (4.15)$$

where the functions $F_{\nu}^{(q)}[f(y)]$ are given by relations (2.35), (2.36) and (2.37). As for the interior components, the quantities (4.15) diverge at the sphere surface, $r = a$. The leading terms of the asymptotic expansions are determined by same formulas (2.44) and (2.45) with the replacement $a - r \rightarrow r - a$.

When $D = 1$ from Eq.(4.15) we obtain the expectation values for the one dimensional semi-infinite region $r > a$. Now the consideration similar to (2.46) and (2.47) yields

$$\varepsilon = -\frac{1}{2\pi} \int_m^{\infty} dz \frac{4\xi z^2 - m^2}{\sqrt{z^2 - m^2}} \frac{\tilde{B}_a z + \tilde{A}_a}{\tilde{B}_a z - \tilde{A}_a} e^{-2z(r-a)}, \quad p = 0, \quad D = 1. \quad (4.16)$$

These results can also be obtained from Eq.(2.47) and (2.46) in the limit $a \rightarrow \infty$, $r + a = \text{const}$ (in these formulas $r + a$ is the distance from the boundary and corresponds to $r - a$ in Eq.(4.16)). For the massless scalar Eq.(4.16) is a special case of the general formula from [41] for a Robin single plate in an arbitrary dimension.

In the case of a massless scalar the asymptotic behavior for expectation values (4.15) at large distances from the sphere can be obtained by introducing a new integration variable $y = zr/a$ and expanding the subintegrand in terms of a/r . The leading contribution for the summand with a given l has order $(a/r)^{2l+2D-1}$ (assuming that $A_a \neq \nu B_a$) and the main contribution comes from the $l = 0$ term. As a result we can see that for $r \gg a$

$$\varepsilon \approx p \approx -\frac{D-2}{D-1} p_{\perp}, \quad (4.17)$$

with

$$\varepsilon \approx \frac{\xi - \xi_c}{8r^3 \ln r/a}, \quad D = 2, \quad (4.18)$$

and

$$\varepsilon \approx (\xi - \xi_c) \frac{\tilde{A}_a(D-2)}{\tilde{A}_a - B_a n} \frac{\Gamma(D-1/2)\Gamma(\frac{D+1}{2})}{2^{D-1}\pi^{D/2}\Gamma^2(D/2)} \frac{a^{D-2}}{r^{2D-1}}, \quad D > 2. \quad (4.19)$$

The case $D = 1$ simply follows from Eq.(4.16). For the conformally coupled scalar the vacuum densities behave as $1/r^{2D+1}$.

In analogy to the interior case the outside subtracted surface energy density can be presented in the form

$$\left\langle T_{00}^{(\text{surf})}(x) \right\rangle_{SUB} = \delta(r - a - 0) \frac{4\xi - 1}{2\pi a S_D r^{D-1}} \sum_{l=0}^{\infty} D_l \int_{ma}^{\infty} \frac{z dz}{\sqrt{z^2 - m^2 a^2}} \frac{\bar{I}_{\nu}^{(a)}(z)}{\bar{K}_{\nu}^{(a)}(z)} F_s[K_{\nu}(zr/a)], \quad (4.20)$$

where $F_s[f(y)]$ is defined as in Eq.(3.11). Integration of this formula gives the total surface energy

$$E_{\text{ext}}^{(\text{surf})} = \frac{4\xi - 1}{2\pi a} \sum_{l=0}^{\infty} D_l \int_{ma}^{\infty} \frac{z dz}{\sqrt{z^2 - m^2 a^2}} \frac{\bar{I}_{\nu}^{(a)}(z)}{\bar{K}_{\nu}^{(a)}(z)} F_s[K_{\nu}(z)], \quad (4.21)$$

localized on the outer surface of the sphere.

Integrating the energy density (4.15), $q = \varepsilon$, over the volume outside a sphere we obtain the corresponding volume energy:

$$E_{\text{ext}}^{(\text{vol})} = \frac{1}{2\pi a} \sum_{l=0}^{\infty} D_l \int_{ma}^{\infty} \frac{z dz}{\sqrt{z^2 - m^2 a^2}} \frac{\bar{I}_{\nu}^{(a)}(z)}{\bar{K}_{\nu}^{(a)}(z)} F_v[K_{\nu}(z)], \quad (4.22)$$

with the notation (3.14). Now the total vacuum energy for the exterior region is obtained as the sum of the volume and surface parts,

$$E_{\text{ext}} = E_{\text{ext}}^{(\text{vol})} + E_{\text{ext}}^{(\text{surf})} = \frac{1}{2\pi a} \sum_{l=0}^{\infty} D_l \int_{ma}^{\infty} \frac{dz}{z} \sqrt{z^2 - m^2 a^2} \frac{\bar{I}_{\nu}^{(a)}(z)}{\bar{K}_{\nu}^{(a)}(z)} F_t[K_{\nu}(z)], \quad (4.23)$$

where $F_t[f(y)]$ is defined in accordance with Eq.(3.17). As we see, as in the interior case, the dependences on the curvature coupling ξ in the surface and volume energies cancel each other.

The radial projection of the force acting per unit area of the sphere from outside is determined as

$$F_{\text{ext}} = -p|_{r=a+} = \frac{1}{2\pi a^{D+1} S_D} \sum_{l=0}^{\infty} D_l \int_{ma}^{\infty} \frac{z^3 dz}{\sqrt{z^2 - m^2 a^2}} \frac{\bar{I}_{\nu}^{(a)}(z)}{\bar{K}_{\nu}^{(a)}(z)} F_{\nu}^{(p)}[K_{\nu}(z)]. \quad (4.24)$$

Using relation (3.19), for the massless scalar field one finds

$$a^D S_D F_{\text{ext}} = E_{\text{ext}}^{(\text{vol})} + \frac{4D}{4\xi - 1} (\xi - \xi_c) E_{\text{ext}}^{(\text{surf})}. \quad (4.25)$$

As in the interior case, this relation can also be obtained directly from the continuity equation (2.33). Recall that the second summand on the right comes from the nonzero trace for the nonconformally coupled massless scalar. The exterior surface, volume, total energies, and vacuum forces acting on the sphere written in the form (4.21)-(4.24) are divergent. Here the corresponding scheme to extract finite results is similar to that for the interior quantities and is explained in section 3.

We now turn to the case of a *spherical shell with zero thickness*. The total vacuum energy including the interior and exterior contributions can be obtained by summing Eqs.(3.15) and (4.22), and the resulting vacuum force by summing Eqs.(3.18) and (4.24):

$$E^{(\text{vol})}(a) = E_{\text{in}}^{(\text{vol})}(a) + E_{\text{ext}}^{(\text{vol})}(a), \quad E(a) = E_{\text{in}}(a) + E_{\text{ext}}(a), \quad F = F_{\text{in}} + F_{\text{ext}}. \quad (4.26)$$

Assuming that for the exterior and interior boundary conditions the coefficients A and B are the same (this corresponds to $B_1 = -B_a$ in Eqs.(2.1) and (4.1)) after some transformations the corresponding expressions can be presented in the forms

$$E^{(\text{surf})}(a) = \frac{4\xi - 1}{2\pi a} \sum_{l=0}^{\infty} D_l \int_{ma}^{\infty} \frac{z dz}{\sqrt{z^2 - m^2 a^2}} \left[1 - \left(\beta + \frac{n}{2} \right) \frac{\left(\tilde{I}_{\nu}(z) \tilde{K}_{\nu}(z) \right)'}{z \tilde{I}_{\nu}'(z) \tilde{K}_{\nu}'(z)} \right] \quad (4.27)$$

for the total surface energy,

$$E(a) = -\frac{1}{2\pi a} \sum_{l=0}^{\infty} D_l \int_{ma}^{\infty} \frac{dz}{z} \sqrt{z^2 - m^2 a^2} \left[2\beta + (\nu^2 - \beta^2 + z^2) \frac{\left(\tilde{I}_{\nu}(z) \tilde{K}_{\nu}(z) \right)'}{z \tilde{I}_{\nu}'(z) \tilde{K}_{\nu}'(z)} \right] \quad (4.28)$$

for the total energy, and

$$F(a) = -\frac{1}{2\pi a^{D+1} S_D} \sum_{l=0}^{\infty} D_l \int_{ma}^{\infty} \frac{z dz}{\sqrt{z^2 - m^2 a^2}} \times \left[2\beta - \xi_1 + \left[\nu^2 - \beta^2 + z^2 + \xi_1 \left(\beta + \frac{n}{2} \right) \right] \frac{(\tilde{I}_\nu(z) \tilde{K}_\nu(z))'}{z \tilde{I}'_\nu(z) \tilde{K}'_\nu(z)} \right] \quad (4.29)$$

for the resulting force per unit area acting on the spherical shell. In these formulas we use the notations

$$\tilde{f}(z) = z^\beta f(z), \quad \beta = A/B, \quad (4.30)$$

for a given function $f(z)$ and ξ_1 is defined in Eq.(2.36). It can be easily seen that in the case $m = 0$ and $\beta = D/2 - 1$ formula (4.28) coincides with the expression for the Casimir energy of the TM modes derived in [10]. The corresponding electromagnetic force can be obtained by differentiating this expression with respect to a .

Using the relation

$$2\beta + (z^2 + \nu^2 - \beta^2) \frac{(\tilde{I}_\nu(z) \tilde{K}_\nu(z))'}{z \tilde{I}'_\nu(z) \tilde{K}'_\nu(z)} = z \frac{(\bar{I}_\nu(z) \bar{K}_\nu(z))'}{\bar{I}_\nu(z) \bar{K}_\nu(z)}, \quad (4.31)$$

the total Casimir energy (4.28) can also be presented in the form

$$E(a) = -\frac{1}{2\pi a} \sum_{l=0}^{\infty} D_l \int_{ma}^{\infty} dz \sqrt{z^2 - m^2 a^2} \frac{(\bar{I}_\nu(z) \bar{K}_\nu(z))'}{\bar{I}_\nu(z) \bar{K}_\nu(z)}. \quad (4.32)$$

For the massless field, using the trace relation (3.21) and integrating the continuity equation (2.33) one obtains

$$E^{\text{vol}} = a^D S_D F - S_D \int_0^\infty dr r^{D-2} \langle T_i^i \rangle_{\text{reg}}. \quad (4.33)$$

Note that though for the massless scalar the subintegrand is a total divergence (see, Eq.(2.5)) the corresponding contribution to the right hand side of Eq.(4.33) is nonzero as the function $(d/dr) \langle \varphi^2 \rangle_{\text{reg}}$ is discontinuous at $r = a$. This contribution can be expressed via the total surface energy, like to Eq.(3.20) or Eq.(4.25).

The formulas for the cases of Dirichlet and Neumann boundary conditions can be obtained taking $B_1 = 0$ ($\beta = \infty$) and $A_1 = 0$ ($\beta = 1 - D/2$), respectively. For example, in the case of Dirichlet vacuum force per unit area one has

$$F(a) = -\frac{1}{2\pi a^{D+1} S_D} \sum_{l=0}^{\infty} D_l \int_{ma}^{\infty} \frac{z dz}{\sqrt{z^2 - m^2 a^2}} \left[4(n+1)\xi - n + z \frac{(I_\nu(z) K_\nu(z))'}{I_\nu(z) K_\nu(z)} \right]. \quad (4.34)$$

By using the expressions for D_l and S_D it can be seen that in the case of a minimally coupled massless scalar ($\xi = 0$, $m = 0$) the expression (4.34) coincides with the formula derived in [9].

For the Dirichlet and Neumann boundary conditions from Eq.(4.27) one has

$$E_{(D)}^{(\text{surf})} = -E_{(N)}^{(\text{surf})} = \frac{1-4\xi}{2\pi} \sum_{l=0}^{\infty} D_l \int_m^\infty \frac{z dz}{\sqrt{z^2 - m^2}}. \quad (4.35)$$

These quantities are independent of the sphere radius and, if we follow the regularization procedure developed in [9, 10], we conclude that for a thin spherical shell the regularized surface

energy vanishes for Dirichlet and Neumann scalars. More generally, for Dirichlet and Neumann boundary conditions the regularized interior, Eq.(3.12), and exterior, Eq.(4.21), surface energies vanish separately. To see this note that VEV $\langle 0|\varphi(x)\partial'_r\varphi(x')|0\rangle$ is zero for $r = a$ in the Dirichlet case and for $r' = a$ in the Neumann case. As a result for these values the subtracted product $\langle\varphi(x)\partial'_r\varphi(x')\rangle_{\text{reg}}$ coincides with the corresponding Minkowskian part $\langle 0_M|\varphi(x)\partial'_r\varphi(x')|0_M\rangle$ (this can also be seen by direct evaluation, applying to the sum over l the Gegenbauer addition theorem, and taking the value of the remained integral from [45]) and is zero in the limit $r' \rightarrow a$ or $r \rightarrow a$ if we regularize the Minkowskian part to be zero. Now we lead to our conclusion if we recall that the surface energy is proportional to $\langle\varphi(a)\partial_r\varphi(a)\rangle_{\text{reg}}$. This is in agreement with the regularization procedure from [9, 10]. First of all, taking into account that the regularized value for the integro-sum on the right of Eq.(4.35) is zero, we can derive from Eqs.(3.12) and (4.21) the relations

$$E_{\text{in(D)}}^{(\text{surf})} = E_{\text{in(N)}}^{(\text{surf})} = -E_{\text{ext(D)}}^{(\text{surf})} = -E_{\text{ext(N)}}^{(\text{surf})}. \quad (4.36)$$

Hence, it is sufficient to consider the Dirichlet interior energy. It can be written in the form

$$\frac{E_{\text{in(D)}}^{(\text{surf})}}{a^D S_D(4\xi - 1)} = \lim_{r \rightarrow a-} \langle\varphi(a)\partial_r\varphi(r)\rangle_{\text{reg}} = - \lim_{r \rightarrow a-} \partial_r \langle 0_M|\varphi(a)\varphi(r)|0_M\rangle, \quad (4.37)$$

where we have used that for a Dirichlet scalar $\langle 0|\varphi(a)\partial_r\varphi(r)|0\rangle = 0$. Now taking into account expression (2.27) for the Minkowskian Wightman function, expanding $K_{(D-1)/2}(m(a-r))$, and using

$$(1-x)^{-\alpha} = \sum_{l=0}^{\infty} x^l \frac{\Gamma(l+\alpha)}{\Gamma(\alpha)l!}, \quad (4.38)$$

for the interior surface energy one obtains

$$E_{\text{in(D)}}^{(\text{surf})} = \frac{4\xi - 1}{2^D a \Gamma(D/2)} \sum_{k=0}^{[(D-1)/2]} \frac{(-1)^k (ma)^{2k}}{k! \Gamma(D/2 - k)} \sum_{l=0}^{\infty} \frac{\Gamma(l + D - 2k)}{l!}, \quad (4.39)$$

where the square brackets denote the largest integer less than or equal to its argument. By using the analytic continuation over D in [9] it has been shown that the regularized value for the sum over l in Eq.(4.39) is zero, and hence this is the case for the Dirichlet surface energy as well.

The method of extracting finite results from expressions (4.27)-(4.29) is the same as that for the interior and exterior quantities and is explained in section 3. In particular, subtracting the leading term in the asymptotic expansion of the subintegrand for large z and using Eq.(3.23), the total Casimir energy may be presented in the form

$$E = -\frac{1}{2\pi a} \sum_{l=0}^{\infty} D_l \int_{ma}^{\infty} dz \sqrt{z^2 - m^2 a^2} \frac{d}{dz} \ln |z^{2\delta_{B0}-1} \bar{I}_\nu(z) \bar{K}_\nu(z)|. \quad (4.40)$$

Here the individual integrals in the series are convergent. This formula can be alternatively derived using, for instance, the Green function method [9, 10] or applying to the corresponding mode sum the methods already used in [11]-[13],[16]-[19]. For the case of a $D = 3$ massless scalar with Neumann boundary condition we recover the known result from [16].

5 Vacuum energy density and stresses in a spherical layer

We have seen that in the intermediate region between two concentric spheres the Wightman function for the scalar field can be presented in the form (4.8), where the term with the first integral is the corresponding unregularized function for the region outside a single sphere with

radius a . The regularization of this term was carried out in the previous section. The second integral on the right of Eq.(4.8) will give a finite result at the coincidence limit for $a \leq r < b$ and needs no regularization. As a result for the regularized Wightman function in the region between two spheres one obtains

$$\begin{aligned} \langle \varphi(x)\varphi(x') \rangle_{\text{reg}} &= \langle \varphi(x)\varphi(x') \rangle_{\text{reg}}^{\text{ext}} - \frac{1}{\pi n S_D} \sum_{l=0}^{\infty} \frac{2l+n}{(rr')^{n/2}} C_l^{n/2}(\cos \theta) \\ &\times \int_m^{\infty} \frac{z dz}{\sqrt{z^2 - m^2}} \Omega_{a\nu}(az, bz) G_{\nu}^{(a)}(az, zr) G_{\nu}^{(a)}(az, zr') \cosh \left[\sqrt{z^2 - m^2}(t' - t) \right], \end{aligned} \quad (5.1)$$

where $\langle \varphi(x)\varphi(x') \rangle_{\text{reg}}^{\text{ext}}$ is the corresponding function for the vacuum outside a single sphere with radius a given by Eq.(4.13), and we have introduced the notation

$$\Omega_{a\nu}(az, bz) = \frac{\bar{K}_{\nu}^{(b)}(bz)/\bar{K}_{\nu}^{(a)}(az)}{\bar{K}_{\nu}^{(a)}(az)\bar{I}_{\nu}^{(b)}(bz) - \bar{K}_{\nu}^{(b)}(bz)\bar{I}_{\nu}^{(a)}(az)}. \quad (5.2)$$

Substituting Eq.(5.1) into Eq.(2.6) we obtain that the vacuum EMT has the diagonal form (2.32) with components

$$q(a, b, r) = q(a, r) + q_a(a, b, r), \quad a < r < b, \quad q = \varepsilon, p, p_{\perp}, \quad (5.3)$$

where $q(a, r)$ are the corresponding functions for the vacuum outside a single sphere with radius a . In Eq.(5.3) the additional components are in the form

$$q_a(a, b, r) = -\frac{1}{2\pi r^n S_D} \sum_{l=0}^{\infty} D_l \int_m^{\infty} \frac{z^3 dz}{\sqrt{z^2 - m^2}} \Omega_{a\nu}(az, bz) F_{\nu}^{(q)} \left[G_{\nu}^{(a)}(az, zr) \right], \quad q = \varepsilon, p, p_{\perp}, \quad (5.4)$$

where $F_{\nu}^{(q)}[f(y)]$ is defined by relations (2.35), (2.36), and (2.37). The quantities (5.4) are finite for $a \leq r < b$ and diverge at the surface $r = b$.

It can be seen that for the case of two spheres the Wightman function in the intermediate region can also be presented in the form

$$\begin{aligned} \langle \varphi(x)\varphi(x') \rangle_{\text{reg}} &= \langle \varphi(x)\varphi(x') \rangle_{\text{reg}}^{\text{in}} - \frac{1}{\pi n S_D} \sum_{l=0}^{\infty} \frac{2l+n}{(rr')^{n/2}} C_l^{n/2}(\cos \theta) \\ &\times \int_m^{\infty} \frac{z dz}{\sqrt{z^2 - m^2}} \Omega_{b\nu}(az, bz) G_{\nu}^{(b)}(bz, zr) G_{\nu}^{(b)}(bz, zr') \cosh \left[\sqrt{z^2 - m^2}(t' - t) \right], \end{aligned} \quad (5.5)$$

with $\langle \varphi(x)\varphi(x') \rangle_{\text{reg}}^{\text{in}}$ being the regularized Wightman function for the vacuum inside a single sphere with radius b (see Eq.(2.28) with replacement $a \rightarrow b$), and

$$\Omega_{b\nu}(az, bz) = \frac{\bar{I}_{\nu}^{(a)}(az)/\bar{I}_{\nu}^{(b)}(bz)}{\bar{K}_{\nu}^{(a)}(az)\bar{I}_{\nu}^{(b)}(bz) - \bar{K}_{\nu}^{(b)}(bz)\bar{I}_{\nu}^{(a)}(az)}. \quad (5.6)$$

Note that in the coincidence limit, $x' = x$, the second summand on the right hand side of Eq.(5.1) is finite on the boundary $r = a$, and is divergent on the boundary $r = b$. Similarly, the second summand on the right of formula (5.5) is finite on the boundary $r = b$ and diverges on the boundary $r = a$. It follows from here that if we write the regularized Wightmann function in the form

$$\langle \varphi(x)\varphi(x') \rangle_{\text{reg}} = \langle \varphi(x)\varphi(x') \rangle_{\text{reg}}^{\text{in}} + \langle \varphi(x)\varphi(x') \rangle_{\text{reg}}^{\text{ext}} + \Delta W(x, x'), \quad (5.7)$$

then in the coincidence limit the "interference" term $\Delta W(x, x')$ is finite for all values $a \leq r \leq b$. Using formulas (2.28) and (5.1) it can be seen that this term may be presented as

$$\Delta W(x, x') = \frac{-1}{\pi n S_D} \sum_{l=0}^{\infty} \frac{2l+n}{(rr')^{n/2}} C_l^{n/2} (\cos \theta) \int_m^{\infty} \frac{z dz}{\sqrt{z^2 - m^2}} W^{(ab)}(r, r') \cosh \left[\sqrt{z^2 - m^2} (t' - t) \right], \quad (5.8)$$

where

$$W^{(ab)}(r, r') = \frac{\bar{I}_\nu^{(a)}(az) \bar{K}_\nu^{(b)}(bz)}{\bar{I}_\nu^{(b)}(bz) \bar{K}_\nu^{(a)}(az)} \left[\frac{G_\nu^{(a)}(az, zr) G_\nu^{(b)}(bz, zr')}{\bar{K}_\nu^{(a)}(az) \bar{I}_\nu^{(b)}(bz) - \bar{K}_\nu^{(b)}(bz) \bar{I}_\nu^{(a)}(az)} - I_\nu(zr') K_\nu(zr) \right]. \quad (5.9)$$

On the basis of formula (5.5) the vacuum EMT components may be written in another equivalent form:

$$q(a, b, r) = q(b, r) + q_b(a, b, r), \quad a < r < b, \quad q = \varepsilon, p, p_\perp, \quad (5.10)$$

with $q(a, r)$ being the corresponding components for the vacuum inside a single sphere with radius b (expressions (2.34) with replacement $a \rightarrow b$). Here the additional components are given by the formula

$$q_b(a, b, r) = -\frac{1}{2\pi r^n S_D} \sum_{l=0}^{\infty} D_l \int_m^{\infty} \frac{z^3 dz}{\sqrt{z^2 - m^2}} \Omega_{b\nu}(az, bz) F_\nu^{(q)} \left[G_\nu^{(b)}(bz, zr) \right], \quad q = \varepsilon, p, p_\perp. \quad (5.11)$$

This expressions are finite for all $a < r \leq b$ and diverge at the inner sphere surface $r = a$.

It follows from the above that if we present the vacuum EMT components in the form

$$q(a, b, r) = q(a, r) + q(b, r) + \Delta q(a, b, r), \quad a < r < b, \quad (5.12)$$

then the quantities (no summation over i)

$$\Delta q(a, b, r) = q_a(a, b, r) - q(b, r) = q_b(a, b, r) - q(a, r) = \hat{\theta}_{ii} \Delta W(x, x') \quad (5.13)$$

are finite for all $a \leq r \leq b$, and $i = 0, 1, 2$ correspond to $q = \varepsilon, p, p_\perp$, respectively. Near the surface $r = a$ it is suitable to use the first equality in Eq.(5.13), as for $r \rightarrow a$ both summands are finite. For the same reason the second equality is suitable for calculations near the outer surface $r = b$. In particular, the additional radial vacuum pressure on the sphere with $r = \alpha$, $\alpha = a, b$ due to the existence of the second sphere ("interaction" force) can be found from Eqs.(5.4) and (5.11), respectively. Using the relations

$$G_\nu^{(r)}(rz, rz) = -B_r, \quad rz \frac{\partial}{\partial y} G_\nu^{(r)}(rz, y) |_{y=rz} = A_r, \quad r = a, b, \quad (5.14)$$

they can be presented in the form

$$\begin{aligned} p_\alpha(a, b, r = \alpha) &= -\frac{1}{2\pi \alpha^D S_D} \sum_{l=0}^{\infty} D_l \int_m^{\infty} \frac{z dz}{\sqrt{z^2 - m^2}} \Omega_{\alpha\nu}(az, bz) \\ &\times \left\{ \tilde{A}_\alpha^2 - 4(D-1)\xi \tilde{A}_\alpha B_\alpha - [z^2 \alpha^2 + l(l+D-2)] B_\alpha^2 \right\}, \quad \alpha = a, b. \end{aligned} \quad (5.15)$$

Unlike the self-action forces this quantity is finite for $a < b$ and needs no further regularization. Using the Wronskian

$$\bar{K}_\nu^{(\alpha)}(\alpha z) \frac{d}{dz} \bar{I}_\nu^{(\alpha)}(bz) - \bar{I}_\nu^{(\alpha)}(\alpha z) \frac{d}{dz} \bar{K}_\nu^{(\alpha)}(bz) = \frac{A_\alpha^2 - B_\alpha^2(z^2 \alpha^2 + \nu^2)}{z} \quad (5.16)$$

it can be seen that

$$[A_\alpha - B_\alpha(z^2\alpha^2 + \nu^2)] \Omega_{\alpha\nu}(az, bz) = -n_\alpha\alpha \frac{\partial}{\partial\alpha} \ln \left| 1 - \frac{\bar{I}_\nu^{(a)}(az)}{\bar{I}_\nu^{(b)}(bz)} \frac{\bar{K}_\nu^{(b)}(bz)}{\bar{K}_\nu^{(a)}(az)} \right|, \quad (5.17)$$

where $n_a = 1$, $n_b = -1$. This allows us to write the expressions (5.15) for the interaction forces per unit surface in another equivalent form:

$$p_\alpha(a, b, r = \alpha) = \frac{n_\alpha}{2\pi\alpha^{D-1}S_D} \sum_{l=0}^{\infty} D_l \int_m^\infty \frac{zdz}{\sqrt{z^2 - m^2}} \left[1 - \frac{\xi_1 \tilde{A}_\alpha B_\alpha}{A_\alpha^2 - B_\alpha^2(z^2\alpha^2 + \nu^2)} \right] \times \frac{\partial}{\partial\alpha} \ln \left| 1 - \frac{\bar{I}_\nu^{(a)}(az)}{\bar{I}_\nu^{(b)}(bz)} \frac{\bar{K}_\nu^{(b)}(bz)}{\bar{K}_\nu^{(a)}(az)} \right|, \quad \alpha = a, b, \quad (5.18)$$

where ξ_1 is defined in Eq.(2.36). For Dirichlet and Neumann scalars the second term in the square brackets is zero.

6 Total Casimir energy in the region between spherical surfaces

In this section we will consider the total Casimir energy in the region between two concentric spherical surfaces. The expression for the unregularized VEV for the energy density can easily be obtained from Eq.(4.10) by applying the corresponding second order operator from Eq.(2.6):

$$\langle 0|T_{00}(x)|0\rangle = \frac{\pi^3}{8a^3 S_D r^n} \sum_{l=0}^{\infty} D_l \sum_{k=1}^{\infty} \frac{\gamma_{\nu,k}^3 T_\nu^{ab}(\gamma_{\nu,k})}{\sqrt{\gamma_{\nu,k}^2 + m^2 a^2}} f_\nu^{(0)}[g_\nu(\gamma_{\nu,k}, \gamma_{\nu,k}r/a)], \quad (6.1)$$

where we use the notation (3.2). The total volume energy in the region between the spheres is obtained by integrating this expression over r . Using the standard integrals involving cylinder functions one finds

$$E_{a < r < b}^{(\text{vol})} = E_{a \leq r \leq b} + \frac{4\xi - 1}{2a} \sum_{l=0}^{\infty} D_l \sum_{k=1}^{\infty} \frac{\gamma_{\nu,k} T_\nu^{ab}(\gamma_{\nu,k})}{\sqrt{\gamma_{\nu,k}^2 + \mu^2}} \left[\tilde{A}_b B_b \frac{\bar{J}_\nu^2(\gamma_{\nu,k})}{\bar{J}_\nu^2(b\gamma_{\nu,k}/a)} - \tilde{A}_a B_a \right], \quad (6.2)$$

where

$$E_{a \leq r \leq b} = \frac{1}{2} \sum_{l=0}^{\infty} D_l \sum_{k=1}^{\infty} \omega_{\nu,k}, \quad \omega_{\nu,k} = \sqrt{\gamma_{\nu,k}^2/a^2 + m^2}, \quad (6.3)$$

is the total Casimir energy for the region between the plates evaluated as a mode sum of the zero-point energies for each normal mode. As in the case for the region inside a sphere, here the energy obtained by integration from the energy density differs from the energy (6.3). This difference can be interpreted as due to the surface energy contribution:

$$E_{ab}^{(\text{surf})} = \frac{4\xi - 1}{2a} \sum_{l=0}^{\infty} D_l \sum_{k=1}^{\infty} \frac{\gamma_{\nu,k} T_\nu^{ab}(\gamma_{\nu,k})}{\sqrt{\gamma_{\nu,k}^2 + \mu^2}} \left[\tilde{A}_b B_b \frac{\bar{J}_\nu^2(\gamma_{\nu,k})}{\bar{J}_\nu^2(b\gamma_{\nu,k}/a)} - \tilde{A}_a B_a \right]. \quad (6.4)$$

To see this, recall that there is an additional contribution to the energy density operator located on the boundaries $r = a, b$ and having the form (3.7). It can be shown that by evaluating the corresponding VEV with the help of Eq.(4.6) and integrating over the region between the spheres we obtain formula (6.4).

To derive expressions for the corresponding subtracted quantities we use the formula

$$\langle T_{00}^{(\text{surf})}(x) \rangle_{SUB} = -(2\xi - 1/2)\delta(x, \partial M)n_r (\partial_r \langle \varphi(r)\varphi(r') \rangle_{SUB})_{r'=r} \quad (6.5)$$

with the subtracted Wightman function from Eq.(5.1), $n_r = 1$ for $r = a + 0$ and $n_r = -1$ for $r = b - 0$. Using this formula for the integrated surface energy on the surfaces $r = a + 0$ (outer surface of the inner sphere) and $r = b - 0$ (inner surface of the outer sphere) one obtains

$$E_{ab}^{(\text{surf})}(r) = n_r \frac{4\xi - 1}{2\pi} \sum_{l=0}^{\infty} D_l \int_m^{\infty} \frac{zdz}{\sqrt{z^2 - m^2}} \times \left\{ \frac{\bar{I}_\nu^{(a)}(za)}{\bar{K}_\nu^{(a)}(za)} F_s[K_\nu(rz)] + \Omega_{a\nu}(az, bz) F_s[G_\nu^{(a)}(az, rz)] \right\}, \quad (6.6)$$

where we use the notation (3.11). With the help of the alternative representation (5.5) for the Wightman function in the region between the spheres we obtain another form of the surface energy,

$$E_{ab}^{(\text{surf})}(r) = n_r \frac{4\xi - 1}{2\pi} \sum_{l=0}^{\infty} D_l \int_m^{\infty} \frac{zdz}{\sqrt{z^2 - m^2}} \times \left\{ \frac{\bar{K}_\nu^{(b)}(zb)}{\bar{I}_\nu^{(b)}(zb)} F_s[I_\nu(rz)] + \Omega_{b\nu}(az, bz) F_s[G_\nu^{(b)}(bz, rz)] \right\}, \quad r = a + 0, b - 0. \quad (6.7)$$

The equivalence of formulas (6.6) and (6.7) can be seen directly using the identities

$$\begin{aligned} \frac{\bar{K}_\nu^{(b)}(zb)}{\bar{I}_\nu^{(b)}(zb)} F_u[I_\nu(rz)] + \Omega_{b\nu}(az, bz) F_u[G_\nu^{(b)}(bz, rz)] &\equiv \frac{\bar{I}_\nu^{(a)}(za)}{\bar{K}_\nu^{(a)}(za)} F_u[K_\nu(rz)] \\ &+ \Omega_{a\nu}(az, bz) F_u[G_\nu^{(a)}(az, rz)] \end{aligned} \quad (6.8)$$

with $u = s, v$ and $r = a, b$. By virtue of relations (5.14) the surface energies (6.6) may be presented in the form

$$E_{ab}^{(\text{surf})}(r) = E^{(\text{surf})}(r) - \frac{n_r}{2\pi} (4\xi - 1) \tilde{A}_r B_r \sum_{l=0}^{\infty} D_l \int_m^{\infty} \frac{zdz}{\sqrt{z^2 - m^2}} \Omega_{r\nu}(az, bz), \quad (6.9)$$

$$r = a + 0, \quad b - 0.$$

The first term on the right corresponds to the surface energy of a single sphere and the second one is induced by the existence of the second sphere and is finite for $a < b$. For Dirichlet and Neumann scalars these surface energies vanish.

Integrating the vacuum energy density (5.3) ($q = \varepsilon$) over the region between the spheres with the help of integration formula (3.13) for the corresponding volume energy, one obtains

$$E_{ab}^{(\text{vol})} = -\frac{1}{2\pi} \sum_{l=0}^{\infty} D_l \int_m^{\infty} \frac{zdz}{\sqrt{z^2 - m^2}} \times \left\{ \frac{\bar{I}_\nu^{(a)}(za)}{\bar{K}_\nu^{(a)}(za)} F_v[K_\nu(rz)] + \Omega_{a\nu}(az, bz) F_v[G_\nu^{(a)}(az, rz)] \right\}_{r=a}^{r=b}. \quad (6.10)$$

Another form is obtained by using the energy density in the form (5.10) (or equivalently identity (6.8)). Replacing the expression in braces at the upper limit $r = b$ in (6.10) with the help of

identity (6.8) the expression for the volume energy in the region $a < r < b$ can be presented in the form

$$E_{ab}^{(\text{vol})} = E_{\text{in}}^{(\text{vol})}(b) + E_{\text{ext}}^{(\text{vol})}(a) + \frac{1}{2\pi} \sum_{l=0}^{\infty} D_l \int_m^{\infty} \frac{z dz}{\sqrt{z^2 - m^2}} \quad (6.11)$$

$$\times \sum_{\alpha=a,b} n_{\alpha} \Omega_{\alpha\nu}(az, bz) \left\{ (4\xi - 1) \tilde{A}_{\alpha} B_{\alpha} + \left(1 - \frac{m^2}{z^2} \right) [A_{\alpha}^2 - (z^2 \alpha^2 + \nu^2) B_{\alpha}^2] \right\}.$$

Now the total Casimir energy in the region between the spheres may be obtained by summing the volume and surface parts,

$$E_{ab} = E_{ab}^{(\text{vol})} + E^{(\text{surf})}(a+0) + E^{(\text{surf})}(b-0) = \quad (6.12)$$

$$= -\frac{1}{2\pi} \sum_{l=0}^{\infty} D_l \int_m^{\infty} \frac{dz}{z} \sqrt{z^2 - m^2} \left\{ \frac{\bar{I}_{\nu}^{(a)}(za)}{\bar{K}_{\nu}^{(a)}(za)} F_t[K_{\nu}(rz)] + \Omega_{a\nu}(az, bz) F_t[G_{\nu}^{(a)}(az, rz)] \right\}_{r=a}^{r=b},$$

where the function $F_t[f(y)]$ is defined as Eq.(3.17). Using identity (6.8), in analogy to (6.11) (or directly summing Eqs.(6.9) and (6.11)), one obtains the formula

$$E_{ab} = E_{\text{in}}(b) + E_{\text{ext}}(a) - \frac{1}{2\pi} \sum_{l=0}^{\infty} D_l \int_m^{\infty} \frac{dz}{z} \sqrt{z^2 - m^2} \quad (6.13)$$

$$\times \sum_{\alpha=a,b} n_{\alpha} \Omega_{\alpha\nu}(az, bz) [A_{\alpha}^2 - (z^2 \alpha^2 + \nu^2) B_{\alpha}^2], \quad n_b = -1, n_a = 1.$$

Note that the last ("interference") term on the right is finite for $a < b$ and needs no further regularization. By virtue of relation (5.17) this formula can also be presented in the form

$$E_{ab} = E_{\text{in}}(b) + E_{\text{ext}}(a) - \frac{1}{2\pi} \sum_{l=0}^{\infty} D_l \int_m^{\infty} dz \sqrt{z^2 - m^2} \frac{d}{dz} \ln \left| 1 - \frac{\bar{I}_{\nu}^{(a)}(az) \bar{K}_{\nu}^{(b)}(bz)}{\bar{I}_{\nu}^{(b)}(bz) \bar{K}_{\nu}^{(a)}(az)} \right|. \quad (6.14)$$

As we might expect the mode sum energy does not depend on the curvature coupling parameter.

Now we turn to a system of two concentric spheres with zero thickness. In addition to the contribution from the intermediate region $a < r < b$ we have to include the contributions of the regions $r < a$ and $r > b$. In these regions the vacuum densities are the same as in the case of a single sphere. For the vacuum force acting per unit area of the sphere with radius $r = \alpha$ one obtains

$$F_{\alpha}(a, b) = F(\alpha) - n_{\alpha} p_{\alpha}(a, b, r = \alpha), \quad \alpha = a, b, \quad n_a = 1, \quad n_b = -1, \quad (6.15)$$

where $F(\alpha)$ is the vacuum force for a single sphere with radius α (see formula (4.29)) and the second summand on the right corresponds to the interaction between the spheres and is given by Eq.(5.15) or Eq.(5.18). Taking into account formula (6.13) we can see that the total vacuum energy is equal to the sum of the vacuum energies for the separate spheres plus an additional "interference" term, given by the third summand on the right of (6.13). Using expression (4.32) for a single shell Casimir energy this leads to

$$E_{0 \leq r \leq \infty}^{(ab)} = -\frac{1}{2\pi} \sum_{l=0}^{\infty} D_l \int_m^{\infty} dz \sqrt{z^2 - m^2} \quad (6.16)$$

$$\times \frac{d}{dz} \ln \left| \bar{I}_{\nu}^{(a)}(az) \bar{K}_{\nu}^{(b)}(bz) \left[\bar{I}_{\nu}^{(b)}(bz) \bar{K}_{\nu}^{(a)}(az) - \bar{I}_{\nu}^{(a)}(az) \bar{K}_{\nu}^{(b)}(bz) \right] \right|.$$

The divergences in this formula are the same as for single spheres with radii a and b , and can be extracted in the way described in section 3.

7 Conclusion

We have considered the Casimir effect for a massive scalar field with general curvature coupling and satisfying the Robin boundary condition on spherically symmetric boundaries in D -dimensional space. Both cases of a single sphere and two concentric spherical surfaces are investigated. All calculations are made at zero temperature and we assume that the boundary conditions are frequency independent. The latter means no dispersive effects are taken into account. The formulation of the theory at finite temperatures can be carried out by using the standard analyticity properties of the finite temperature Green function (see, for instance, [40]). Unlike most previous studies of the scalar Casimir effect here we adopt the local approach. To obtain the expectation values for the energy-momentum tensor we first construct the positive frequency Wightmann function (note that the Wightmann function is also important in considerations of the response of a particle detector at a given state of motion). The application of the generalized Abel-Plana formula to the mode sum over zeros of the corresponding combinations of the cylinder functions allows us to extract the Minkowskian part and to present the subtracted part in terms of exponentially convergent integrals. In this paper we consider a flat background spacetime and the subtraction of the Minkowskian part gives finite results at any strictly interior or exterior points in the coincidence limit. The regularized expectation values for the EMT are obtained by applying on the subtracted part a certain second-order differential operator and taking the coincidence limit. These quantities diverge as the boundary is approached. Surface divergences are well known in quantum field theory with boundaries and are investigated near an arbitrary shaped smooth boundary. They lead to divergent global quantities, such as the total energy or vacuum forces acting on the sphere. To regularize them and to extract numerical results we can apply a procedure based on analytic continuation in the dimension already used in [9, 10] and briefly described in section 3.

The expectation values of the EMT for the region inside a spherical shell are given by formulas (2.34). These expressions are finite at interior points and diverge on the sphere surface. The leading term in the corresponding asymptotic expansion is the same as for the flat boundary case and is zero for a conformally coupled scalar. The coefficients for the subleading asymptotic terms will depend on the boundary curvature, Robin coefficient, and mass. The global quantities for the sphere interior region are considered in section 3. Integrating the unregularized vacuum energy density we show that the volume energy differs from the total vacuum energy, evaluated as the sum of the zero-point energies for the elementary oscillators. We argue that this difference is due to the additional surface contribution to the vacuum energy located on the inner surface of the sphere. We give the expressions for the subtracted volume and surface energies, Eqs.(3.15) and (3.12), and for the force acting per unit area of the sphere from inside, Eq.(3.18). In section 4 first we consider the scalar vacuum in the spherical layer between two surfaces. The quantities characterizing the vacuum outside a single sphere are obtained from this case in the limit when the radius of the outer sphere tends to infinity. Subtracting the parts corresponding to the space without boundaries, for the regularized vacuum densities we derive formulas (4.15). These formulas differ from the ones for the interior region by the replacements $I_\nu \rightarrow K_\nu$, $K_\nu \rightarrow I_\nu$. We also consider the corresponding global quantities for the outer region. As in the interior case there is an additional surface energy, located on the outer surface of the sphere. The decomposition of the total outside Casimir energy into volume and surface parts is provided. The regularized surface energy is zero for Dirichlet and Neumann boundary conditions. In this section we further consider a spherical shell with zero thickness and with the same Robin coefficients on the inner and outer surfaces. The expressions for the total and surface energies, including the contributions from the inside and outside regions, are obtained (formulas (4.28) and (4.27)). The resulting force acting per unit area of the spherical shell is also derived (formula (4.29)). In section 5 we consider the expectation values for the energy-momentum tensor in the

region between two spherical surfaces. The general case is investigated when the constants A_i , B_i , $i = a, b$, in the Robin boundary condition are different for the outer and inner spheres. The expression for the total energy in this region (including the surface parts) is derived. The corresponding regularized Wightmann function can be presented in different equivalent forms (Eqs.(5.1), (5.5) or (5.7)). Note that in the last formula the "interference" term $\Delta W(x, x')$ defined by Eqs.(5.8), (5.9) is finite in the coincidence limit for all values $a \leq r \leq b$. The surface divergences are contained in the two first summands on the right of Eq.(5.7), corresponding to the interior and exterior Wightmann functions of a single sphere. The components of the vacuum EMT may be written in the forms (5.3), (5.10) or (5.12). The additional (interaction) force acting per unit area of the sphere with radius $r = \alpha$, $\alpha = a, b$ due to the existence of the second sphere is given by formula (5.15) or by (5.18) and is finite for $a < b$. In section 6 we consider the total Casimir energy and its decomposition into surface and volume parts. First of all we show that in the general Robin case the unregularized intermediate Casimir energy obtained by integration of the corresponding energy density differs from the vacuum energy evaluated as the sum of the zero-point energies for each normal mode of frequency. As in the single sphere case this difference may be interpreted as due to the surface energy contribution. The corresponding subtracted quantities can be presented in the form (6.9), where the second summand on the right is due to the existence of the second sphere and is finite. The analogous formulas for the integrated volume energy and total Casimir energy in the intermediate region are in the forms (6.11) and (6.13). The latter can also be presented as Eq.(6.14). As in the single sphere case the surface contributions to the vacuum energy vanish for Dirichlet and Neumann scalars. Then we consider a system of two concentric spheres with zero thickness. The vacuum force acting on each sphere can be presented as the sum of the force acting on a single sphere plus an additional interaction force which is due to the existence of the second sphere. The latter is given by formula (5.15) or (5.18) and is finite for $a < b$. Using the expression for a single sphere vacuum energy the total Casimir energy can also be presented in the form (6.16). For a $D = 1$ massless scalar the results given in this paper are special cases of the general formulas derived in [41] for parallel plate geometry in arbitrary dimensions.

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A Appendix: Summation formula over zeros of $C_\nu^{ab}(\lambda, z)$

Here we will consider the series over zeros of the function

$$C_\nu^{ab}(\lambda, z) \equiv \bar{J}_\nu^{(a)}(z)\bar{Y}_\nu^{(b)}(\eta z) - \bar{Y}_\nu^{(a)}(z)\bar{J}_\nu^{(b)}(\eta z), \quad (\text{A.1})$$

where the quantities with overbars are defined in accordance with Eq.(4.3). This type of series arises in calculations of the vacuum expectation values in confined regions with boundaries of spherical form. To obtain a summation formula for these series we will use the generalized Abel-Plana formula (GAPF) [31, 32]. In this formula as functions $g(z)$ and $f(z)$ let us substitute

$$g(z) = \frac{1}{2i} \left[\frac{\bar{H}_\nu^{(1b)}(\eta z)}{\bar{H}_\nu^{(1a)}(z)} + \frac{\bar{H}_\nu^{(2b)}(\eta z)}{\bar{H}_\nu^{(2a)}(z)} \right] \frac{h(z)}{C_\nu^{ab}(\eta, z)}, \quad f(z) = \frac{h(z)}{\bar{H}_\nu^{(1a)}(z)\bar{H}_\nu^{(2a)}(z)}, \quad (\text{A.2})$$

where for definiteness we shall assume that $\eta > 1$ and the notations $\bar{F}^{(i)}$, $i = a, b$, are introduced in accordance with Eq.(4.3). The sum and difference of these functions are

$$g(z) - (-1)^k f(z) = -i \frac{\bar{H}_\nu^{(ka)}(\lambda z)}{\bar{H}_\nu^{(ka)}(z)} \frac{h(z)}{C_\nu^{ab}(\eta, z)}, \quad k = 1, 2. \quad (\text{A.3})$$

The condition for the generalized Abel-Plana formula written in terms of the function $h(z)$ is as follows:

$$|h(z)| < \varepsilon_1(x) e^{c_1|y|} \quad |z| \rightarrow \infty, \quad z = x + iy, \quad (\text{A.4})$$

where $c_1 < 2(\eta - 1)$, $x^{\delta_{Ba0} + \delta_{Bb0} - 1} \varepsilon_1(x) \rightarrow 0$ for $x \rightarrow +\infty$. Let $\gamma_{\nu,k}$ be zeros of the function $C_\nu^{ab}(\eta, z)$ in the right half plane. In this section we will assume values of ν , A , and B for which all these zeros are real and simple, and the function $\bar{H}_\nu^{(1a)}(z)$ ($\bar{H}_\nu^{(2a)}(z)$) has no zeros in the right half of the upper (lower) half-plane. For real ν and A_a, A_b, B_a, B_b the zeros $\gamma_{\nu,k}$ are simple. To see this note that the function $J_\nu(tz) \bar{Y}_\nu^{(a)}(z) - Y_\nu(tz) \bar{J}_\nu^{(a)}(z)$ is a cylinder function with respect to t . Using the standard result for indefinite integrals containing the product of any two cylinder functions (see [43, 44]) it can be seen that

$$\int_1^\eta t \left[J_\nu(tz) \bar{Y}_\nu^{(a)}(z) - Y_\nu(tz) \bar{J}_\nu^{(a)}(z) \right]^2 dt = \frac{2}{\pi^2 z T_\nu^{ab}(\eta, z)}, \quad z = \gamma_{\nu,k}, \quad (\text{A.5})$$

where we have introduced the notation

$$T_\nu^{ab}(\eta, z) = z \left\{ \frac{\bar{J}_\nu^{(a)2}(z)}{\bar{J}_\nu^{(b)2}(\eta z)} [A_b^2 + B_b^2(\eta^2 z^2 - \nu^2)] - A_a^2 - B_a^2(z^2 - \nu^2) \right\}^{-1}. \quad (\text{A.6})$$

On the other hand

$$\frac{\partial}{\partial z} C_\nu^{ab}(\eta, z) = \frac{2}{\pi T_\nu^{ab}(\eta, z)} \frac{\bar{J}_\nu^{(b)}(\eta z)}{\bar{J}_\nu^{(a)}(z)}, \quad z = \gamma_{\nu,k}. \quad (\text{A.7})$$

Combining the last two results we deduce that for real ν , A_α, B_α the derivative (A.7) is nonzero and hence the zeros $z = \gamma_{\nu,k}$ are simple. By using this it can be seen that

$$\text{Res}_{z=\gamma_{\nu,k}} g(z) = \frac{\pi}{2i} T_\nu^{ab}(\eta, \gamma_{\nu,k}) h(\gamma_{\nu,k}). \quad (\text{A.8})$$

Let $h(z)$ be an analytic function for $\text{Re} z \geq 0$ except the poles $z_k (\neq \gamma_{\nu i})$, $\text{Re} z_k > 0$, and with a possible branch point at $z = 0$. (For the physical application in this paper (see section 5) the corresponding function is analytic. However to keep the formula general we will consider the case of a meromorphic function.) By using the GAPF, in analogy with the derivation of the summation formula (4.13) in [32], it can be seen that if it satisfies condition (A.4) and

$$h(ze^{\pi i}) = -h(z) + o(z^{-1}), \quad z \rightarrow 0, \quad (\text{A.9})$$

and the integral

$$\text{p.v.} \int_0^b \frac{h(x) dx}{\bar{J}_\nu^{(a)2}(x) + \bar{Y}_\nu^{(a)2}(x)} \quad (\text{A.10})$$

exists, then

$$\begin{aligned} & \lim_{b \rightarrow +\infty} \left\{ \frac{\pi^2}{2} \sum_{k=1}^m h(\gamma_{\nu,k}) T_\nu^{ab}(\eta, \gamma_{\nu,k}) + r_\nu [h(z)] - \text{p.v.} \int_0^b \frac{h(x) dx}{\bar{J}_\nu^{(a)2}(x) + \bar{Y}_\nu^{(a)2}(x)} \right\} = \\ & = -\frac{\pi}{2} \text{Res}_{z=0} \left[\frac{h(z) \bar{H}_\nu^{(1b)}(\eta z)}{C_\nu^{ab}(\eta, z) \bar{H}_\nu^{(1a)}(z)} \right] - \frac{\pi}{4} \int_0^\infty \frac{\bar{K}_\nu^{(b)}(\eta x)}{\bar{K}_\nu^{(a)}(x)} \frac{[h(xe^{\pi i/2}) + h(xe^{-\pi i/2})] dx}{\bar{K}_\nu^{(a)}(x) \bar{I}_\nu^{(b)}(\eta x) - \bar{K}_\nu^{(b)}(\eta x) \bar{I}_\nu^{(a)}(x)}, \end{aligned} \quad (\text{A.11})$$

where $\gamma_{\nu,m} < b < \gamma_{\nu,m+1}$. Here the functional $r_\nu[h(z)]$ is defined as

$$\begin{aligned} r_\nu[h(z)] = & \pi \sum_k \text{Res}_{\text{Im}z_k=0} \left[\frac{\bar{J}_\nu^{(a)}(z) \bar{J}_\nu^{(b)}(\eta z) + \bar{Y}_\nu^{(a)}(z) \bar{Y}_\nu^{(b)}(\eta z)}{\bar{J}_\nu^{(a)2}(z) + \bar{Y}_\nu^{(a)2}(z)} \frac{h(z)}{C_\nu^{ab}(\eta, z)} \right] \\ & + \pi \sum_{k,l=1,2} \text{Res}_{(-1)^l \text{Im}z_k < 0} \left[\frac{\bar{H}_\nu^{(lb)}(\eta z)}{\bar{H}_\nu^{(la)}(z)} \frac{h(z)}{C_\nu^{ab}(\eta, z)} \right]. \end{aligned} \quad (\text{A.12})$$

In section 5 formula (A.11) is used to derive the regularized vacuum energy-momentum tensor in the intermediate region between two spherical shells. Note that the corresponding functions $h(z)$ are analytic and hence $r_\nu[h(z)] = 0$ for them. In the case $A_a = A_b$, $B_a = B_b$ from Eq.(A.11) one obtains the summation formula (4.13) in [32].

Note that Eq.(A.11) may be generalized for the functions $h(z)$ with purely imaginary poles $\pm i y_k$, $y_k > 0$, satisfying the condition

$$h(z e^{\pi i}) = -h(z) + o((z \mp i y_k)^{-1}), \quad z \rightarrow \pm i y_k, \quad (\text{A.13})$$

and in the case of the existence of purely imaginary zeros $\pm i y_{\nu,k}$, $y_{\nu,k} > 0$ for $C_\nu^{ab}(\eta, z)$. The corresponding formula is obtained from Eq.(A.11) by adding to the right hand side residue terms for $z = i y_k$, $i y_{\nu,k}$ in the form

$$- \pi \sum_k \text{Res}_{z=\eta_k} \left[\frac{\bar{H}_\nu^{(1b)}(\eta z)}{\bar{H}_\nu^{(1a)}(z)} \frac{h(z)}{C_\nu^{ab}(\eta, z)} \right], \quad \eta_k = i y_k, i y_{\nu,k}, \quad (\text{A.14})$$

and taking the principal value of the integral on the right which exists by virtue of Eq.(A.13). The arguments here are similar to those for the Remark after Theorem 3 in [32].

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